THE CALCULUS

FOR BEGINNERS

CAMBRIDGE UNIVERSITY PRESS
London FETTER LANE, E.C.
C. F. CLAY, MANAGER

Edinburgh: 100, PRINCES STREET
Berlin: A. ASHER AND CO.
Leipzig: F. A. BROCKHAUS
New York: G. P. PUTNAM'S SONS
Bombay and Calcutta: MACMILLAN AND Co., LTD.
Toronto: J. M. DENT AND SONS, LTD.
Tokyo: THE MARUZEN-KABUSHIKI-KAISHA

THE CALCULUS

FOR BEGINNERS

by

John William

J. W. MERCER, M.A.

Head of the Mathematical Department, Royal Naval College, Dartmouth

Author of *Trigonometry for beginners*

Cambridge:

at the University Press

1914

First Edition, 1910.
Reprinted, 1912, 1914.

PREFACE

IN writing this short course, I have been guided by my conviction that it is much more important for the beginner to understand clearly what the processes of the Calculus mean, and what it can do for him, than to acquire facility in performing its operations or a wide acquaintance with them. I had much rather that a boy, confronted with a problem, should, after analysing it, be able to say "If I could differentiate (or integrate) this function of x, I could solve the problem" than that he should be able to perform the operation without seeing its bearing on the problem.

The book is intended primarily for those who are or will be interested in the applications of the Calculus to Physics and Engineering.

I believe that boys, especially on the Science side of a School, could profitably begin the study of the subject at a much earlier age than is usual. I believe, too, that even those who are to be Mathematicians would do well to go through some such course as is here indicated before proceeding to the more formal treatises, for comparatively young students gain by approaching a subject inductively before studying it deductively. Even if they are possessed of mathematical ability, it is good that they should have a thorough comprehension of the goal in particular cases

before undertaking general proofs, and should be reminded by a preliminary course depending on more laborious methods, of the advantage of replacing "mere counting" by mathematical processes.

As I have already indicated, no great prominence is given to methods of differentiating and integrating complicated expressions, in fact, x^n is the only function whose differential coefficient is required in the first 250 pages. Later, $\sin x$, $\cos x$, etc., e^x and $\log x$ are dealt with, and the Engineer needs little beyond these.

Before reaching "$\frac{dy}{dx}$," I have devoted many pages—some may think too many—to the ideas of rate of change and of the limiting value of the ratio of two continually diminishing quantities. Unless the ground is cleared in this way, a boy is apt to begin and continue indefinitely his study of the subject in a state of mental fog. I do not believe that anything is gained by hurrying over the early stages. On the contrary, I believe that the ultimate result of reducing the time spent on them is a very great loss of power.

When x^n has been differentiated, the result is applied to problems of maxima and minima, geometrical properties of curves, approximations, etc. Then the inverse problem "Given $\frac{dy}{dx}$, find y" is discussed, and this leads up to the Integral Calculus and its application to Areas, Volumes, Work, etc. These should not be looked upon as so many detached problems each requiring its own particular rule, but if the meaning of integration is properly understood, they will all be seen to depend on one general principle. This will explain why I have given so much space to the discussion of a few typical problems ab initio.

After this, rules are investigated for differentiating

$\sin x$, $\cos x$, $\tan x$, etc., e^x, $\log_e x$, products, quotients, etc., and more problems are dealt with of the same nature as before but requiring a knowledge of these rules.

Chapters XII.—XIV. form a sort of appendix with short notices of the application of the Calculus to the Solution of Equations, Methods of Integration and Polar Co-ordinates.

No great amount of previous mathematical knowledge is assumed. A boy is supposed to know his Elementary Algebra and Trigonometry and to have some slight acquaintance with the co-ordinate geometry of the straight line. He should be able to write down the equation of a line through a given point with a given gradient and should know the relation between the gradients of perpendicular lines. He is also expected to have had some practice in drawing graphs from their equations and to know what these graphs mean. Many of the illustrations given are arithmetical, algebraical and geometrical, but in addition to these many will be found drawn from the scientific knowledge usually at the command of boys of this age.

Several of the examples are intended to lead up to the ideas afterwards presented in the text, and in the more difficult ones, especially in some which will be found in the miscellaneous sets, the boy of mathematical ability will find plenty of scope for his skill in the manipulation of symbols.

It would probably be well to omit the section on Points of Inflexion (pp. 127—135) on a first reading It is found there, because there did not seem to be a more convenient place for it.

I take this opportunity of thanking Professor Hobson of Cambridge for some notes on the earlier chapters, which he sent to me after seeing the manuscript, but this must not be taken to mean that anything I have written has his sanction.

I am very grateful also to Mr A. W. Siddons of Harrow, for some criticisms and suggestions which he made after wading through most of the original manuscript.

The subject has been taught here during the last few years to boys of 16 on the lines of this book and I am much indebted to my colleagues who have helped in various ways during the passage of the book through the press.

Thanks are also due to Professor Lamb of Manchester for permission to reprint two tables from his "Infinitesimal Calculus," to Professor Ewing, Director of Naval Education, for permission to use questions set to Naval Officers, and to the Controller of His Majesty's Stationery Office for permission to include some questions set at Examinations by the Board of Education and the Civil Service Commissioners.

J. W. MERCER.

ROYAL NAVAL COLLEGE,
DARTMOUTH,
June 1910.

CONTENTS

CHAPTER I

FUNDAMENTAL NOTIONS

CHAPTER II

DIFFERENTIATION FROM FIRST PRINCIPLES

CHAPTER III

DIFFERENTIATION OF x^n

CHAPTER IV

MAXIMA AND MINIMA

CHAPTER V

SMALL ERRORS AND APPROXIMATIONS

CHAPTER VI

THE INVERSE OPERATION

CHAPTER VII

INTEGRAL CALCULUS. AREAS OF PLANE CURVES. MEAN ORDINATE

CHAPTER VIII

FURTHER APPLICATIONS OF THE INTEGRAL CALCULUS

CHAPTER IX

DIFFERENTIATION OF TRIGONOMETRICAL RATIOS

CHAPTER X

HARDER DIFFERENTIATION

CHAPTER XI

DIFFERENTIATION OF n^x

CHAPTER XII

APPROXIMATE SOLUTION OF EQUATIONS

CHAPTER XIII

METHODS OF INTEGRATION

CHAPTER XIV

POLAR CO-ORDINATES

CHAPTER I

FUNDAMENTAL NOTIONS

Speed

1. Uniform speed. To measure the speed of a moving body, we must know the distance it travels in some definite time. The speed is said to be **uniform** if equal distances are travelled in equal times, however small those equal times may be. Thus if a train has a uniform speed of 60 miles an hour, it will travel 30 miles in every half hour, 10 miles in every 10 mins., 1 mile in every min., 88 ft. in every sec., 8·8 ft. in every ·1 sec., and so on, and we could state the speed as 60 miles/hour, or 1 mile/minute, or 88 ft./sec., etc.

We can deduce any one of these expressions for the speed from any one of the relations between distance and time:

e.g. 60 miles an hour is 60×5280 ft. in 60×60 secs.

$$\text{or } \frac{60 \times 5280}{60 \times 60} \text{ ft. in 1 sec.}$$

or 88 ft./sec.,

$$8{\cdot}8 \text{ ft. in } {\cdot}1 \text{ sec. is } \frac{8{\cdot}8}{{\cdot}1} \text{ ft. in 1 sec.}$$

or 88 ft./sec.,

and generally if a body moving with uniform speed travels s ft. in an interval of t secs., $\frac{s}{t}$ will have the same value whatever be

the duration of the interval and at whatever stage of the journey it be taken, and in this case $\frac{s}{t}$ ft./sec. is said to be the speed of the body at every instant.

2. Average speed. Suppose a train travels 60 miles in a certain hour, but that we do not know that its speed has been uniform during the hour, we can say that its **average speed** during the hour is 88 ft./sec., meaning that if it kept up a uniform speed of 88 ft./sec. for an hour, i.e. travelled 88 feet in every separate second, the distance covered in the hour would be 60 miles, the same as the distance actually covered. Actually it might cover 70 ft. in one second, 90 ft. in another, and so on, provided that the total distance was 60 miles.

If a body travels 3 ft. in $\frac{1}{4}$ sec., we say that its average speed during this quarter of a second is 12 ft./sec., meaning that if a body had a uniform speed of 12 ft./sec., it would in $\frac{1}{4}$ sec. travel 3 ft, the same as the distance actually travelled.

The distance travelled in any other quarter of a second would not as a rule be 3 feet, unless the motion were uniform; suppose for example that in the next quarter of a second the distance travelled were 5 feet, then the average speed during this next quarter of a second would be 20 ft./sec. and during the complete half second 16 ft./sec.

Generally, if a body travels s ft. in t secs., its average speed during this interval of t secs. is

$$\frac{s}{t} \text{ ft./sec.,}$$

and as a rule the value of $\frac{s}{t}$ will depend on the duration of the interval and on the stage of the journey at which it is taken.

EXERCISES. I.

1. A train leaves Bristol at 4.0 p.m. and arrives at Exeter, $75\frac{1}{3}$ miles distant, at 5.55 p.m. Find its average speed (i) in miles per hour, (ii) in feet per second.

2. A train travels 40 miles in the first hour, 35 in the second, 45 in the third, 50 in the fourth and 40 in the fifth. Find in miles per hour its average speed for the whole journey.

3. A stone falls 1 ft. in $\frac{1}{4}$ sec. Find its average speed during this interval (i) in miles per hour, (ii) in yards per minute.

4. A body travels ·0035 ins. in ·02 of a second. Find in ft./sec. its average speed during this interval.

5. A body travels ·00063 ft. in ·001 of a second. Find in miles/hour its average speed during this interval.

3. Speed at an instant. Suppose we are studying the motion of some moving body and that we are able to measure the distances it travels in certain small intervals of time after passing some fixed mark. In our first experiment we will suppose that we get the distance travelled in one second. The next experiment we will suppose to be made with improved apparatus which will enable us to find the distance travelled in half a second, and so on, each experiment being a more refined one than its predecessor.

Tabulate the results of the experiments as follows:

	Time in secs.	*Distance in ins.*
(1)	1	48
(2)	$\frac{1}{2}$	24
(3)	$\frac{1}{4}$	12
(4)	$\frac{1}{8}$	6
(5)	$\frac{1}{16}$	3

In this case we should probably conclude that the body was moving with a uniform speed of 48 ins./sec.

If however the results were

(1)	1	48
(2)	$\frac{1}{2}$	21
(3)	$\frac{1}{4}$	9·75
(4)	$\frac{1}{8}$	4·69
(5)	$\frac{1}{16}$	2·30

it is obvious that the speed is not uniform.

What shall we say then is the speed of the body as it passes the fixed point?

The average speeds during the intervals used in the different experiments are as shewn in the following table:

	Interval in secs.	*Average speed in ins./sec.*
(1)	1	48
(2)	$\frac{1}{2}$	42
(3)	$\frac{1}{4}$	39
(4)	$\frac{1}{8}$	37·52
(5)	$\frac{1}{16}$	36·80

These results are all different. Now suppose we took the result of the first experiment and assumed that the body moved for the whole second with a uniform speed of 48 ins./sec. and from this calculated how far it would go in $\frac{1}{16}$ sec., we should get 3 ins., which is by no means the same as 2·30 ins., the actual distance travelled. If, however, we used the result of the second experiment, and assumed that the body moved for half a second

with a uniform speed of 42 ins./sec., and calculated from this the distance it would go in $\frac{1}{16}$ sec., we should get 2·625 ins., and in the same way if we assumed uniform speeds of 39 and 37·52 ins./sec., we should get 2·44 and 2·34 ins. respectively.

These distances 3, 2·625, 2·44, 2·34 ins. approximate more and more to the true distance 2·3 ins. If we tried to deduce the distance travelled in $\frac{1}{32}$ sec. we should get a better approximation by using the average speed 36·8 ins./sec. than by using any of the others, but the result 1·15 ins. would not be correct. Still, if the apparatus at our disposal did not permit us to measure the distance travelled in any shorter period of time than $\frac{1}{16}$ sec., we should content ourselves by saying that the speed of the body as it passed the given point was approximately 36·8 ins./sec.

If we could improve our apparatus and measure the distance travelled in $\frac{1}{100}$ sec., say ·361 ins., we should say that the speed of the body as it passed the given point was still more nearly 36·1 ins./sec. If from this we calculated the distance travelled in any shorter interval, say $\frac{1}{1000}$ sec., our result ·0361 ins. would be nearer the truth than a result obtained by using any of the other average speeds.

But however refined our measurements, we could never by a calculation of this kind obtain the speed of the body as it passed the given point. We calculate each time the **average speed** of the body during a short interval following the instant in question and the effect of improving our apparatus is that we are able to shorten the interval for which the average speed is calculated. We cannot say that the body moves for any interval of time, however short, with uniform speed, but the shorter the interval of time for which we make this assumption the smaller is the error made in the calculated distance for a still shorter interval.

EXERCISES. II.

1. Three men A, B, C begin to observe the motion of a train just as it reaches a quarter-mile post. A finds that in the next second it goes 44 ft., B that it takes $4\frac{1}{2}$ secs. for the whole train (210 ft. long) to pass him, C that it takes 26 secs. to reach the next quarter-mile post. Find in ft./sec. what each would obtain for the average speed of the train. What is the best available result for the speed of the train as it passes the first post? If each assumes that the speed he obtains is uniform over the time considered, and uses it to calculate the distance travelled by the train in the next half second after leaving the post, what would their results be, and which would be nearest to the truth? [Nearest tenth of a foot.]

2. A train leaves the terminus A and passes through the stations B, C, D, etc. at the times indicated in the table. The second column gives distances from A. [80 chains = 1 mile.]

	m.	*ch.*	*Time of passing*
A	0	00	5.20
B	1	$43\frac{1}{2}$	5.24
C	2	41	5.26
D	3	$31\frac{3}{4}$	$5.27\frac{1}{4}$
E	4	04	$5.28\frac{1}{4}$
F	4	$78\frac{1}{4}$	$5.29\frac{1}{2}$
G	6	$34\frac{1}{2}$	$5.31\frac{1}{2}$
H	8	$29\frac{1}{2}$	5.34
I	9	12	5.35

Find the average speed between the stations in chains (nearest chain) per minute. What would you guess to be the approximate speed as the train passed through (i) F, (ii) H?

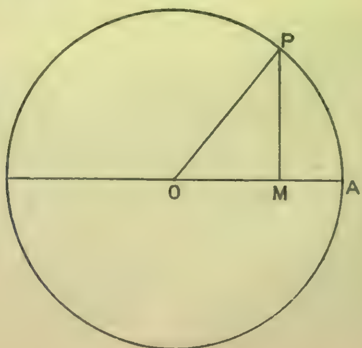


Fig. 1.

3. A point P starts from A (Fig. 1) and moves round a circle of radius 100 cms. with uniform speed, making a complete revolution in 1 min. PM is perpendicular to OA. What is OM when $\angle$ AOP is (i) 60°, (ii) 55°, (iii) 65°, (iv) 57°, (v) 63°, (vi) 59°, (vii) 61°? What is the average speed (cms./sec.) of M as $\angle$ AOP increases (i) from 55° to 60°, (ii) from 60° to 65°, (iii) from 57° to 60°, (iv) from 60° to 63°, (v) from 59° to 60°, (vi) from 60° to 61°? What is approximately the speed of M when $\angle$ AOP = 60°?

4. A point P starts from A and moves round a circle of radius 100 cms. with uniform speed, making a complete revolution in 1 min. OP meets the tangent at A in T. What is AT when $\angle$ AOP is (i) 60°, (ii) 55°, (iii) 65°, (iv) 57°, (v) 63°, (vi) 59°, (vii) 61°? What is the average speed (cms./sec.) of T as $\angle$ AOP increases (i) from 55° to 60°, (ii) from 60° to 65°, (iii) from 57° to 60°, (iv) from 60° to 63°, (v) from 59° to 60°, (vi) from 60° to 61°? What is approximately the speed of T when $\angle$ AOP = 60°?

4. Now suppose the distance travelled in a given time is not determined by experiment but given by an algebraical formula, say

$$s = 5 + 3t + 2t^3,$$

s ft. being the distance from some fixed point at the end of t secs. measured from some fixed instant, and suppose we want the speed at the end of 4 secs.

If in the formula we put $t = 4$, we get $s = 145$; i.e. the distance from the fixed point at the end of 4 secs. from the fixed instant is 145 ft.

Similarly the distance from the fixed point at the end of 5 secs. from the fixed instant is 270 ft.

Reckoning from the end of the fourth second, i.e. from the instant at which we are to find the speed, we see that the distance travelled in the next second is 125 ft. and the average speed during this second is 125 ft./sec. Now calculate the distance travelled in $4\frac{1}{2}$ secs. from the beginning. It is $200\frac{3}{4}$ ft., and the distance described in the $\frac{1}{2}$ sec. after the end of the fourth second is $55\frac{3}{4}$ ft. $\therefore$ the average speed during this $\frac{1}{2}$ sec. is $55\frac{3}{4} \div \frac{1}{2}$ or $111\frac{1}{2}$ ft./sec.

Similarly find the distances and average speeds for ·2, ·1, ·01,

·001, ·0001 of a second after the end of the fourth second. The results should be as tabulated.

Time in secs.	*Distance in ft.*	*Average speed in ft./sec.*
1	125	125
·5	55·75	111·5
·2	20·776	103·88
·1	10·142	101·42
·01	0·992402	99·2402
·001	0·099024002	99·024002
·0001	0·009900240002	99·00240002

This table could be continued to any extent, and it appears that the average speed as we diminish the interval of time over which it is calculated is always greater than 99 ft./sec., approaches more and more to 99 ft./sec., and apparently could be made as near to 99 ft./sec. as we like if we extend the table far enough, i.e. if we make the interval of time small enough.

To make quite sure of this let us consider the distance travelled in $(4+h)$ secs., where h can eventually be as small a fraction as we please. The distance is

$$5+3(4+h)+2(4+h)^3 \text{ ft. or } 145+99h+24h^2+2h^3 \text{ ft.,}$$

i.e. the distance travelled in the interval h secs. after the end of the fourth second is $(99h+24h^2+2h^3)$ ft. $\therefore$ the average speed during this interval is

$$\frac{99h+24h^2+2h^3}{h} \text{ or } (99+24h+2h^2) \text{ ft./sec.}$$

[Notice that this result includes all the results in the table, e.g. if we put $h=\cdot 1$ we get $99+2{\cdot}4+{\cdot}02$ or $101{\cdot}42$.]

As h is made smaller and smaller $(24h+2h^2)$ becomes smaller and smaller and can be made as small as we please if h be made small enough. In other words the average speed during the interval h can be made as near to 99 ft./sec. as we please if we take the interval short enough.

This, so to speak, ideal speed of 99 ft./sec. is called the speed of the body at the end of 4 secs.

As this is an extremely important idea, it may be well to summarise the steps by which we arrived at our result.

We found the distance travelled by the body in a short interval immediately following the end of the fourth second and calculated the average speed during this interval. We then did the same thing with a shorter interval, then with a shorter, and so on, and found that our average speeds continually approached 99 ft./sec. and that if we took our interval short enough, we could make the average speed come as near to 99 ft./sec. as we wished.

This limiting value of the average speed which can be approached as nearly as we please by taking the interval short enough, is called the speed of the body at the end of 4 secs.

EXERCISES. III.

1. If $s=5+3t+2t^3$, find the average speed during the interval h secs. which immediately precedes the end of the fourth second. Shew that it is less than 99 ft./sec., continually approaches 99 ft./sec. as h is made smaller and can be brought as near to 99 ft./sec. as we please if we make h small enough.

Also find the average speed during the interval of h seconds which extends from $\frac{h}{2}$ secs. before to $\frac{h}{2}$ secs. after the end of the fourth second, and shew that this average speed also can be brought as near to 99 ft./sec. as we please if we make h small enough.

2. If $s=5t+3t^2$ find the distances travelled in 3, 4, 3·5, 3·2, 3·1, 3·01, 3·001 secs. from the beginning. Hence find the average speeds during the intervals 1, ·5, ·2, ·1, ·01, ·001 secs. immediately following the end of the third second. Also find the distance travelled in $(3+h)$ secs. and the average speed during the interval h secs. after the end of the third second.

What is the speed at the end of the third second?

Find also the average speed during (i) the interval of h seconds immediately preceding the end of the third second, (ii) during the interval of h seconds which extends from the end of $\left(3-\frac{h}{2}\right)$ secs. to the end of $\left(3+\frac{h}{2}\right)$ secs., (iii) during the interval between the end of $(3-m)$ secs. and the end of $(3+n)$ secs., and shew in each case that by making the interval short enough we can bring the average speed as near as we please to 23 ft./sec.

3. In **Qu. 2** find (i) the average speed during the fourth second, (ii) the speeds at the end of 3, 4, $3\frac{1}{2}$ secs., (iii) the mean (half the sum) of the speeds at the end of 3 and 4 secs.

4. If $s=5+9t^3$, find (i) the average speed during the fourth second, (ii) the speed at the end of $3\frac{1}{2}$ secs., (iii) the mean of the speeds at the end of 3 and 4 secs.

5. If $s=t^4-3t+5$, find the distance gone in 4 secs. and in $(4+h)$ secs. Hence get the speed at the end of the fourth second.

5. From Exs. 1 and 2 we see that the speed at a given instant may be deduced from the average speed during a short interval, (i) immediately following, or (ii) immediately preceding, or (iii) including, the instant in question and may be defined as that speed which any of these average speeds can be made to approach as nearly as we please if the interval be made short enough. See § 28.

Rate of Increase.

6. The question of speed might be presented somewhat differently, as shewn in the following examples.

Suppose a train to travel uniformly between two stations distant 15 and 17 miles from the terminus, and to pass those stations at 5.20 and 5.23.

Let s feet be the distance of the train from some fixed point of the path, say the terminus, at the end of t secs. from some fixed instant, say 5 o'clock.

Then as t increases from 20×60 to 23×60

s ,, ,, 15×5280 to 17×5280,

i.e. as t increases by 3×60, s increases by 2×5280,

$\therefore$,, ,, ,, 1 ,, ,, $\dfrac{2 \times 5280}{3 \times 60}$ or $58\frac{2}{3}$,

or $58\frac{2}{3}$ is the rate of increase of s per unit increase of t.

[With our earlier terminology we should say the speed was $58\frac{2}{3}$ ft./sec.]

Notice that this $58\frac{2}{3}$ is $\dfrac{\text{Increase in } s}{\text{Increase in } t}$.

7. It will avoid confusion if we remember in cases like this where we are dealing with quantities of two different kinds, viz.:—distance and time, that s and t are numbers, s being the number of feet in the distance and t the number of seconds in the time.

When we say $\dfrac{\text{Increase in } s}{\text{Increase in } t} = 58\frac{2}{3}$, we are stating, not that $58\frac{2}{3}$ is the result of dividing 2×5280 feet by 3×60 seconds, but that it is the result of dividing 2×5280 by 3×60; 2×5280 being the number of feet and 3×60 the number of seconds.

The statement in § 2 that if a body travels s feet in t seconds its average speed during the interval is $\dfrac{s}{t}$ ft./sec. is equivalent to this: If the number of feet in the distance travelled be divided by the number of seconds in the time taken, the resulting number is the number of ft./sec. in the average speed.

8. If the motion is not uniform, i.e. if the ratio $\dfrac{\text{Increase in } s}{\text{Increase in } t}$ is not constant, as in § 4, we might state our results as follows:

When $t = 4$, $s = 145$.

When t is increased by $\cdot 5$, s is increased by $55{\cdot}75$.

The ratio $\dfrac{\text{Increase in } s}{\text{Increase in } t} = \dfrac{55{\cdot}75}{\cdot 5} = 111{\cdot}50$,

and we say that **111·50** is the **average rate of increase of s per unit increase of t between t = 4 and t = 4·5**; for if the body were moving uniformly in such a way that

an increase of ·5 in t produced an increase of 55·75 in s,

then " " 1 " would produce " 111·50 in s.

Similarly 101·42 is the average rate of increase of s per unit increase of t between $t=4$ and $t=4·1$, for if s were increased by 10·142 in **every** ·1 sec. s would be increased by 101·42 in 1 sec. and so on.

$99+24h+2h^2$ is the average rate of increase of s per unit increase of t between $t=4$ and $t=4+h$ and **99** is said to be the **rate of increase of s per unit increase of t when t = 4.**

Or we might say

111·50 is the **average rate of increase of s with respect to t between t = 4 and t = 4·5**, meaning that as t increases from 4 to 4·5, s increases 111·50 times as much, and **99** is the **rate of increase of s with respect to t when t = 4**, this 99 being the number towards which our succession of average rates continually tends.

9. Suppose the formula connecting s and t to **be**

$$s = 20t - 2t^2,$$

and to fix our ideas we will suppose that s is measured to the right.

If $t=4$, $s=48$: if $t=4+h$, $s=48+4h-2h^2$ and proceeding as before we find that the speed at the end of 4 seconds is 4 ft./sec. to the right.

If $t=5$, $s=50$: if $t=5+h$, $s=50-2h^2$. If we go through the usual steps we say that in the interval of h seconds which follows the end of the 5th second, the body moves $-2h^2$ ft. to the right and its average speed in the interval is **− 2h ft./sec.**

The significance of the negative sign is easily seen. At the end of 5 seconds, the distance of the body to the right of the

starting point is 50 feet: at the end of $(5+h)$ seconds it is $(50-2h^2)$ ft., i.e. in the interval of h secs. the body has moved $2h^2$ ft. to the left and the average speed is **2h ft./sec.** to the left.

If $t=5-h$, $s=50-2h^2$ and in the interval h secs. before the end of the 5th second, the body has moved $2h^2$ ft. to the right and its average speed is **2h ft./sec.** to the right.

This average speed can be made as small as we like if h is made small enough, i.e. if the interval be made short enough.

During any interval, however short, immediately before the end of the 5th second the body is moving to the right and during any interval however small immediately after the end of the 5th second, it is moving to the left and in each case the average speed during the interval can be made as small as we please if we make the interval short enough. We say then, that **at the end of 5 seconds the speed of the body is zero** and that the direction of motion changes at the end of 5 seconds.

10. If $t=7$, $s=42$: if $t=7+h$, $s=42-8h-2h^2$.

Proceeding as before we say that in the interval of h seconds following the end of the 7th second the body moves $-8h-2h^2$ ft. to the right and its average speed in that direction is $(-8-2h)$ ft./sec. and the speed at the end of 7 secs. is -8 ft./sec.

The negative sign is again easily explained.

At the end of 7 secs. the body is 42 ft. to the right of the starting point. At the end of $(7+h)$ secs. it is $42-8h-2h^2$ ft. to the right, i.e. in the interval of h secs. it has moved $(8h+2h^2)$ ft. to the left, the average speed in this direction being $(8+2h)$ ft./sec. and the speed at the end of 7 secs. being 8 ft./sec. to the left.

A negative speed is thus seen to indicate that the body is moving in the opposite direction to that in which s is measured.

Or we might say

As t increases from 7 to $7+h$, s changes from 42 to $42-8h-2h^2$.

Or, as t increases by h, s increases by $-8h-2h^2$

or decreases by $8h+2h^2$,

and the average rate of increase of s per unit increase of t is $-8-2h$,

or the average rate of decrease of s per unit increase of t is $8+2h$.

These two statements mean the same thing; in fact, **to say that the rate of increase of s with respect to t is negative simply means that s is decreasing as t is increasing.**

11. Function. When two quantities are connected in such a way that to each value of the first there corresponds a definite value of the second, the second quantity is said to be a function of the first,

e.g. in the case considered in § 4, s is a function of t; the time being given we can calculate the distance of the body from the fixed point. The relation between s and t is shewn in the form of an equation

$$s=5+3t+2t^3,$$

[In § 3 the relation was exhibited by means of a table of pairs of corresponding values.]

and we have seen how to find

(i) The average rate of increase of s per unit increase of t or the average rate of increase of s with respect to t between two specified values of t. $\left[\text{It is } \dfrac{\text{Increase in } s}{\text{Increase in } t}.\right]$

(ii) The rate of increase of s per unit increase of t or the rate of increase of s with respect to t for a specified value of t.

[It is the value which the ratio $\dfrac{\text{Increase in } s}{\text{Increase in } t}$ can be made to approach as nearly as we please if the increase in t be made small enough.]

12. The area of a square is a function of the length of the side. If A sq. ins. is the area of a square whose side is x ins.

$$A = x^2.$$

Now suppose we have a square plate whose side is 1″ and that it is expanding under the action of heat so that it remains a square.

The original area is 1 sq. in., i.e. the original value of $A = 1$. If the side be increased to 1·1″ the area increases to 1·21 ins., i.e. if x be increased by ·1, A is increased by ·21,

$$\therefore \frac{\text{Increase in A}}{\text{Increase in } x} = \frac{\cdot 21}{\cdot 1} = 2\cdot 1 \text{ (see § 7),}$$

and this is the average rate of increase of A per unit increase of x between $x = 1$ and $x = 1\cdot 1$.

Similarly if the side be increased from 1″ to 1·001″ the area is increased from 1 sq. in. to 1·002001 sq. ins., and the average rate of increase of A per unit increase of x between $x = 1$ and $x = 1\cdot 001$ is 2·001.

If the increase in the side be made smaller and smaller this average rate of increase can be brought as near to 2 as we please,

for if x is increased from 1 to $1 + h$, A is increased from 1 to $1 + 2h + h^2$

and
$$\frac{\text{Increase in A}}{\text{Increase in } x} = \frac{2h + h^2}{h} = 2 + h,$$

which can be made as near to 2 as we please if h is made small enough.

And we say that **when x = 1, the rate of increase of A per unit increase of x, or the rate of increase of A with respect to x, is 2.**

13. If the plate be supposed to be cooling and contracting we shall find that

if x is diminished by ·1, A is diminished by ·19, or we might say

if x is increased by $-\cdot 1$, A is increased by $-\cdot 19$, so that

$$\frac{\text{Decrease in A}}{\text{Decrease in } x} = \frac{\cdot 19}{\cdot 1} = 1\cdot 9,$$

or we might say

$$\frac{\text{Increase in A}}{\text{Increase in } x} = \frac{-\cdot 19}{-\cdot 1} = 1\cdot 9.$$

Similarly if x changes from 1 to $1-h$

A „ „ 1 to $1-2h+h^2$,

and $$\frac{\text{Decrease in A}}{\text{Decrease in } x} = \frac{2h-h^2}{h} = 2-h,$$

or $$\frac{\text{Increase in A}}{\text{Increase in } x} = \frac{-2h+h^2}{-h} = 2-h,$$

and as before, when $x=1$, the rate of increase of A with respect to x is 2.

14. It thus appears that **if the rate of increase of one quantity with respect to another is positive, the two quantities are increasing or decreasing together, but if the rate is negative, one is increasing while the other is decreasing.**

15. The pressure of a given mass of gas at constant temperature is a function of the volume.

Let V c. ins. denote the volume of a given mass of gas and p lbs. per sq. in. its pressure, and suppose the following table given

p	30	40	48	50
V	40	30	25	24

As V decreases from 40 to 24, p increases from 30 to 50,

i.e. $$\frac{\text{Increase of } p}{\text{Decrease of V}} = \frac{20}{16} = \frac{5}{4},$$

or $\frac{5}{4}$ is the average rate of increase of p per unit decrease of V between $V = 40$ and $V = 24$.

Similarly

$\frac{8}{5}$ is the average rate of increase of p per unit decrease of V

or $-\frac{8}{5}$ is the average rate of increase of p with respect to V between $V = 30$ and $V = 25$.

16. Now suppose the relation between p and V given by the formula $pV = 1200$.

We can by calculation form the following table:

V	40	45	42	41	40·5	$40+h$
p	30	26·67	28·57	29·27	29·63	$\frac{1200}{40+h}$

and the average rates of increase of p with respect to V between

$$\left.\begin{array}{ll} \text{(i)} & V=40 \text{ and } V=45 \\ \text{(ii)} & V=40 \text{ and } V=42 \\ \text{(iii)} & V=40 \text{ and } V=41 \\ \text{(iv)} & V=40 \text{ and } V=40\cdot5 \end{array}\right\}$$

will be found to be respectively $-\cdot67$, $-\cdot71$, $-\cdot73$, $-\cdot74$ and what we call the rate of increase of p with respect to V when $V = 40$ is the quantity towards which this succession of numbers continually tends as the increase in V is made smaller.

To make quite certain what this limit is, find the average rate of increase of p with respect to V between $V = 40$ and $V = 40 + h$. It is $-\frac{30}{40+h}$; and by making h small enough we can make this as near $-\frac{30}{40}$ or $-\cdot75$ as we like.

Thus $-\cdot75$ is the rate at which p is increasing with respect to V when $V = 40$.

EXERCISES. IV.

1. In the following table A sq. ins. is the area of a segment of height h ins. in a circle of diameter 1000 ins.

h	100	101	102	103	104
A	40875	41477	42081	42687	43296

Find the average rate of increase of A with respect to h between the following values of h:

(i) 100 and 104; (ii) 100 and 103; (iii) 100 and 102; (iv) 100 and 101.

2. Use tables to find the average rate of increase of $\sin\theta$ with respect to θ (θ being the number of radians in the angle) (i) between $\theta = \cdot 5061$ and $\theta = \cdot 5236$, (ii) between $\theta = \cdot 5236$ and $\theta = \cdot 5411$.

With 4-figure tables what is the best value you can get for the rate of increase of $\sin\theta$ with respect to θ when $\theta = \frac{\pi}{6}$?

3. A square is expanding in such a way that it remains a square. Find the rate of increase of the number of sq. ins. in the area per unit increase of the number of sq. ins. in the side when the side of the square is (i) 3″, (ii) a″.

4. If a cubical block expands so as to remain cubical, find the rate of increase of the number of cubic inches in the volume with respect to the number of inches in the edge when the edge is

(i) 1″; (ii) 5″; (iii) a″.

5. If a circle expands so as to remain circular, and if the radius, circumference and area are respectively r ins., C ins. and A sq. ins., find the rates of increase of

(i) A with respect to r,
(ii) C ,, ,, r,
(iii) A ,, ,, C,

when $r = 3$.

6. The radius of a spherical bubble is increasing at the rate of 1 cm. per second. At a certain instant its radius is 4 cms. What will its radius be 1, ·5, ·2, ·1, ·01, h secs. after this? In each case find the surface and volume.

Find during each interval

(i) the average rate of increase of the number of cubic cms. in the volume with respect to the number of cms. in the radius;

(ii) the average rate of increase of the number of sq. cms. in the surface with respect to the number of cms. in the radius;

(iii) the average increase per sec. of the volume;

(iv) " " " " surface.

Also find at the end of 4 secs.

(v) the rate of increase of the number of c.c. in the volume per unit increase of the number of cms. in the radius;

(vi) the rate of increase of the number of sq. cms. in the surface per unit increase of the number of cms. in the radius;

(vii) the rate of increase of the volume per sec.;

(viii) " " " surface " .

7. The speed of a body (v ft./sec.) at the end of t secs. is given by the formula $\mathbf{v=3+4t^2}$.

Find the average rate of increase of v with respect to t between

(i) $t=2$ and $t=3$; (ii) $t=2$ and $t=2+h$,

and deduce the rate of increase of v with respect to t when $t=2$.

[Note. The first result is 20 and this means that in 1 sec. the speed has increased 20 ft./sec. or that the average acceleration is 20 ft./sec^2. The last result gives the acceleration at the end of 2 secs.]

Gradient.

17. Uniform gradient. Fig. 2 shows a portion of a straight line KL.

P, Q are any two points on the line.

PM, QN are the ordinates of P, Q and PR is parallel to OX.

The $\triangle$ PQR has always the same shape wherever we take P and Q. In particular the ratio $\frac{RQ}{PR}$ is constant.

This ratio is called the **gradient** of the line.

It is the tangent of the angle RPQ or the tangent of the angle which KL makes with OX.

18. We may suppose that we pass from P to Q by two steps, (i) a horizontal step PR, (ii) a vertical step RQ. The gradient is

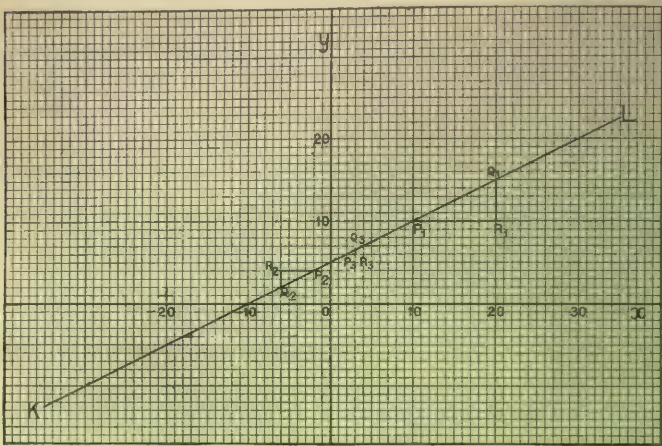


Fig. 2.

the ratio of the vertical step to the horizontal step, and is the same whatever the length of the horizontal step and from whatever point we start.

Following the usual convention we call a horizontal step positive if it is to the right and a vertical step positive if it is upwards; so that the gradient is positive if both steps are positive or both negative, and the gradient is negative if the steps are of opposite signs.

e.g. in Fig. 2

Horizontal step $P_1R_1 = +10$, Vertical step $R_1Q_1 = +5$,

" " $P_2R_2 = -4$, " " $R_2Q_2 = -2$,

and the gradient of the line is $+\frac{1}{2}$.

In Fig. 3

Horizontal step $P_1R_1 = +10$, Vertical step $R_1Q_1 = -15$,

" " $P_2R_2 = -4$, " " $R_2Q_2 = +6$,

and the gradient of the line is $-\frac{3}{2}$.

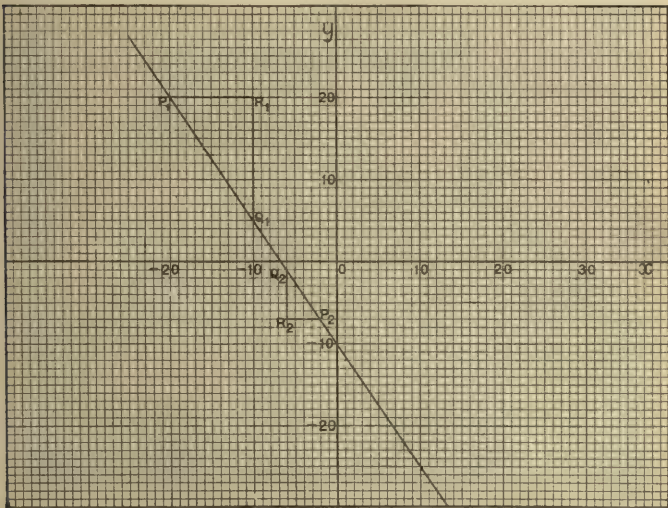


Fig. 3.

19. Expressing this in the language of coordinates we may say that if (x_1y_1) (x_2y_2) be the coordinates of any two points on the line, the gradient is $\frac{y_2 - y_1}{x_2 - x_1}$ this being a fraction which has the same value whatever the coordinates of the points chosen.

e.g. in Fig. 2

coordinates of P_1 are (10, 10) and of Q_1 (20, 15),

and gradient $= \frac{15-10}{20-10} = \frac{1}{2}$;

coordinates of P_2 are (− 2, 4) and of Q_2 (− 6, 2),

and gradient $= \frac{2-4}{-6+2} = \frac{1}{2}$.

20. Another way of looking at the question is as follows:

In Fig. 2 if we take a horizontal step $+1$ [P_3R_3], the vertical step is $+\frac{1}{2}$ [R_3Q_3]; so that we may say that the gradient gives the vertical ascent per unit length of horizontal advance, or the increase of y per unit increase of x.

21. *Example.* Shew that with our definition the gradient of $y = mx + n$ is m.

Let (x_1y_1) (x_2y_2) be any two points on the line,

$$\therefore\ y_2 = mx_2 + n,$$

$$\text{and } y_1 = mx_1 + n;$$

$$\therefore\ y_2 - y_1 = m(x_2 - x_1),$$

$$\therefore\ \frac{y_2 - y_1}{x_2 - x_1} = m,$$

or m is the gradient of the line.

22. Average gradient. If P, Q be two points on a path which is not straight as in Fig. 4, $\frac{RQ}{PR}$ is called the **average gradient** of the path between P and Q; e.g. the average gradient between P and Q is $\frac{6}{9}$. This is the gradient of the line PQ, and we may say that if we move from P along the path so as to make a horizontal step PR, the vertical step RQ is the same as if we had moved along the line PQ which has a uniform gradient $\frac{6}{9}$.

The average gradient will depend on the length of the horizontal step and on the point from which we start, e.g. the average gradient between P and Q_1 is $\frac{16}{19}$ and between P_2 and Q_2 it is $\frac{9{\cdot}5}{9}$.

Generally if $(x_1 y_1)$ $(x_2 y_2)$ be the coordinates of two points P, Q on a curve the average gradient between P and Q is $\frac{y_2 - y_1}{x_2 - x_1}$ and this will be different for different positions of P and Q, i.e. for different values of the coordinates.

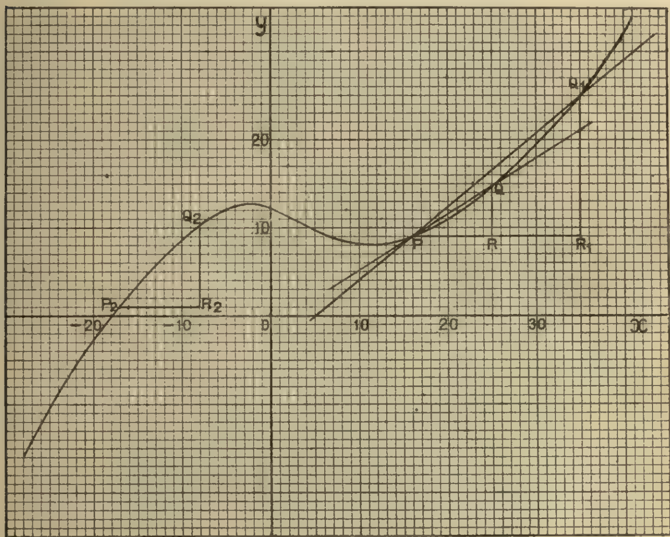


Fig. 4.

Gradient at a point of a curve.

23. Figure 5 shews a portion of the curve

$$y = x^2.$$

P is the point (1, 1), Q_1 is (3, 9).

Let PQ_1 be joined and produced indefinitely both ways. Draw the ordinates PM, Q_1N_1, and draw PR_1 parallel to OX.

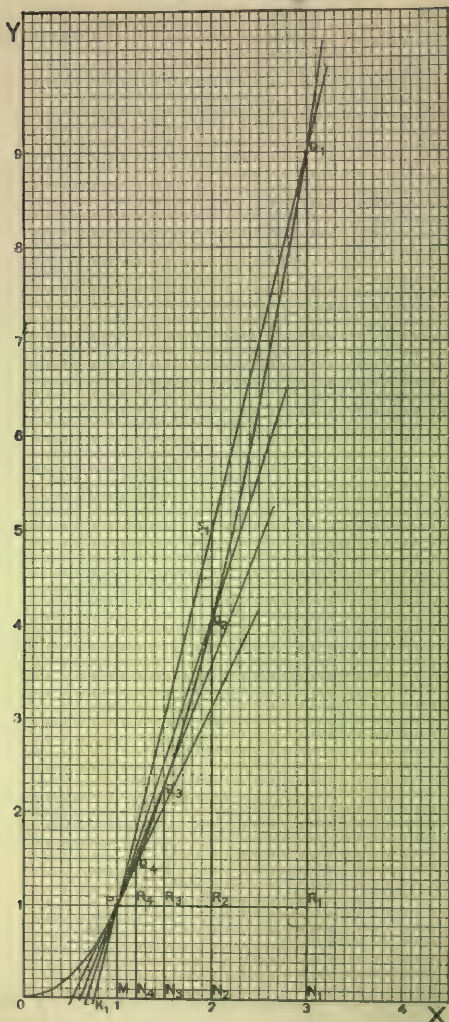


Fig. 5.

Then as x increases from 1 to 3,

y „ „ 1 to 9,

i.e. $\frac{R_1Q_1}{PR_1} = \frac{\text{Increase in } y}{\text{Increase in } x} = \frac{8}{2} = 4 =$ average rate of increase of y per unit increase in x between $x = 1$ and $x = 3$.

This is $\tan R_1PQ_1$ or $\tan XK_1P$ or the tangent of the angle which PQ_1 makes with OX and is called the **gradient of the chord PQ_1.**

[Notice. If $PR_2 = 1$ and R_2S_1 be drawn parallel to OY meeting PQ_1 in S_1, then $R_2S_1 = 4$.]

Now suppose the line PQ_1 to rotate about P until it cuts the curve in Q_2 corresponding to $x = 2$.

We shall get in an exactly similar way $\tan R_2PQ_2$ or $\tan XK_2P$ or the gradient of PQ_2 or the average rate of increase of y per unit increase of x between $x = 1$ and $x = 2$ is $\frac{3}{1}$ or 3.

Let the line continue to rotate and cut the curve successively in Q_3, Q_4, Q_5, Q_6 corresponding to $x = 1{\cdot}4$, $1{\cdot}2$, $1{\cdot}1$, $1{\cdot}01$, and we get

Gradient of PQ_3 or average rate of increase of y per unit increase of x between $x = 1$ and $x = 1{\cdot}4$ is **2·4**.

Gradient of PQ_4 or average rate of increase of y per unit increase of x between $x = 1$ and $x = 1{\cdot}2$ is **2·2**.

Gradient of PQ_5 or average rate of increase of y per unit increase of x between $x = 1$ and $x = 1{\cdot}1$ is **2·1**.

Gradient of PQ_6 or average rate of increase of y per unit increase of x between $x = 1$ and $x = 1{\cdot}01$ is **2·01**.

If we continue this process making the increase in x and therefore at the same time the increase in y smaller and smaller or bringing the point Q nearer and nearer to P, the gradient of the rotating line comes gradually nearer to 2, is always greater than 2, but can be made as near to 2 as we please by bringing Q near enough to P.

To make quite sure of this let the line cut the curve in Q for which $x = 1 + h$ and $\therefore\ y = 1 + 2h + h^2$ (Fig. 6).

Then gradient of $\text{PQ} = \dfrac{\text{Increase in } y}{\text{Increase in } x} = \dfrac{2h + h^2}{h} = 2 + h$,

and by making h small enough we can make this as near 2 as we like.

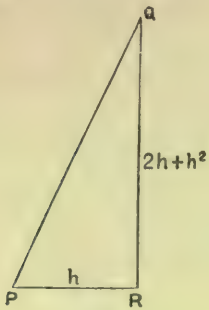


Fig. 6.

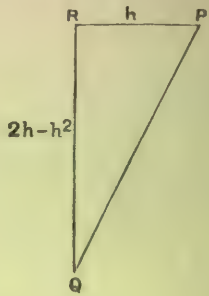


Fig. 7.

If we continue the rotation, the line will cut the curve again on the other side of P, say in a point for which $x = 1 - h$ and $\therefore\ y = 1 - 2h + h^2$ (Fig. 7).

Then gradient of $\text{PQ} = \dfrac{\text{Decrease in } y}{\text{Decrease in } x} = \dfrac{2h - h^2}{h} = 2 - h$,

i.e. the gradient of a chord joining P to a point Q for which $x < 1$ is less than 2, but can be made as near to 2 as we like, by bringing Q near enough to P.

Thus if a chord through P cut the curve in a point to the right of P, however near to P, its gradient > 2; if the chord cut the curve in a point to the left of P, its gradient < 2 but in either case the gradient can be made as near to 2 as we please by bringing the second point near enough to P.

The **gradient of the curve at P is said to be 2.**

It is the same thing that we have previously called the rate of increase of y per unit increase of x, when $x = 1$.

24. The line through P (PT) whose gradient is 2 is called the **tangent at** P.

This line does not cut the curve again in a point near P, for if it did its gradient would not be 2.

In fact PT is not one of the series of chords PQ_1, PQ_2, PQ_3, etc. for we have seen that none of these chords has a gradient 2, but by bringing Q near enough to P we can bring PQ as near to coincidence with PT as we like, i.e. we can make the angle TPQ as small as we like.

This is expressed shortly by saying that the tangent PT is the limiting form of the chord PQ when Q is brought indefinitely near to P.

If PT (Fig. 8) is the tangent at P and PR = 1 then if RT be drawn parallel to OY

$$RT = 2.$$

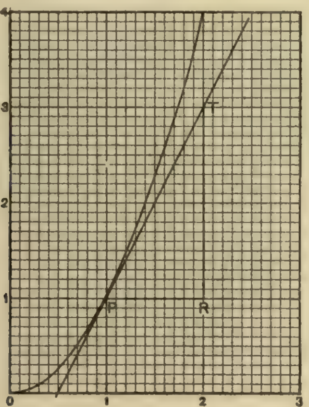


Fig. 8.

25. In these figures we have taken the same scale along both axes and what we have called the gradient of a line is the tangent of the angle which it makes with OX.

e.g. the gradient of PQ_1 is $\dfrac{\text{Actual length of } R_1Q_1}{\text{Actual length of } PR_1}$ or $\tan PK_1X$.

If the scales are not the same the gradient, being

$$\frac{\text{Increase in } y}{\text{Increase in } x} \text{ is } \frac{\text{Number of } y\text{-units in } R_1Q_1}{\text{Number of } x\text{-units in } PR_1},$$

and this is not the tangent of PK_1X.

e.g. if $y = x^2$ be drawn with the scales shewn in Fig. 9, the gradient may be written $\dfrac{R_1Q_1}{PR_1}$ provided it is understood that R_1Q_1 is to be measured by means of the scale marked on OY, and PR_1 by means of the scale marked on OX.

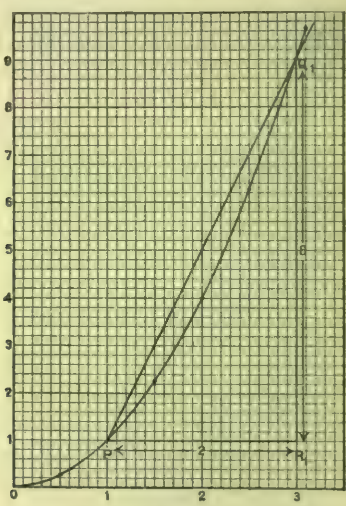


Fig. 9.

Thus gradient $= \dfrac{8}{2} = 4$ as before but now 4 is not $\tan R_1PQ_1$ in Fig. 9. It is the tangent of the angle that we should get in place of R_1PQ_1 if we altered the scales so that the x-unit and the y-unit were of the same length, i.e. it is $\tan R_1PQ_1$ in Fig. 5.

We may look upon Fig. 9 as a deformed copy of Fig. 5 and consider either that horizontal distances are correctly represented and vertical distances shortened in the ratio 1 to 2 or that vertical distances are correctly represented and horizontal distances lengthened in the ratio 2 to 1.

The tangent of the actual angle R_1PQ_1 that we see in Fig. 9 is not 4 but 2.

EXERCISES. V.

1. If $y=x^2$, find y when $x=3$, 3·5, 3·1, 3·01, $3+h$, 2·9, $3-h$. Hence get the gradients of the chords joining the point on the curve $y=x^2$ at which $x=3$ to the points at which $x=3{\cdot}5$, 3·1, 3·01, $3+h$, 2·9, $3-h$.

What is the gradient of the curve at the point (3, 9)?

Draw the graph of $y=x^2$ between $x=1$ and $x=4$ [taking $\frac{1}{2}''$ as the unit for both x and y]. Through (3, 9) draw the line whose gradient is 6. It should be a tangent to the curve. What angle does it make with OX?

2. What is the gradient of the chord joining the points where $x=3-h$ and $x=3+h$. Shew that by making h smaller and smaller we can bring this chord as near as we please to coincidence with the tangent at (3, 9).

3. Draw $y=x^2$ between $x=1$ and $x=4$ taking 1″ as the unit along OX and ·2″ as the unit along OY. As in **Qu. 1** draw through (3, 9) the line whose gradient is 6. It should be a tangent. What angle does it make with OX?

4. Find the gradient of the chord joining the point on the curve $y=x^3$ at which $x=2$ to the point at which $x=2+h$.

Hence get the gradient at the point (2, 8).

5. Find the gradient of the chord joining the points where $x=2-h$ and $x=2+h$ on the curve $y=x^3$. Shew that by making h smaller and smaller we can bring the chord as near as we please to coincidence with the tangent at (2, 8). Similarly for the chord joining the points where $x=2-m$ and $x=2+n$.

6. Find the gradient of $y=x^3$ at the points where $x=0$, 1, 3, 4.

7. In the curve $y=x^3$ you have found the values of y at the points where $x=0$, 1, 2, 3, 4 and also the gradients at these points. Make use

of these ten facts to draw the curve from $x=0$ to $x=4$. Take 1″ as the x-unit and ·1″ as the y-unit. What angle does the tangent at (4, 64) make with OX?

8. Find the gradient of the curve $y=2x+3x^2$ at the points where $x=1,\ -\frac{1}{3}$.

9. Find the gradient of $y=4x+3$ at the points where $x=0, 1, 2$.

10. If $y=x^3(4-x)$ find the values of y when $x=1$ and when $x=1+h$.

Use your result to find "the gradient of the curve $y=x^3(4-x)$ at the point where $x=1$," explaining carefully what the phrase means. Similarly find the gradient where $x=3$.

Using these results and the values of y when $x=0, 1, 2, 3, 4$ draw the graph between $x=0$ and $x=4$.

26. In curves with which we shall deal, the limiting position of the chord PQ when Q is brought indefinitely near to P will be the same on whichever side of P we take Q, and will be the

Fig. 10.

same as the limiting position of the chord RS (P lying between R and S) when R and S are each brought indefinitely near to P. [See Figs. 10, 11.]

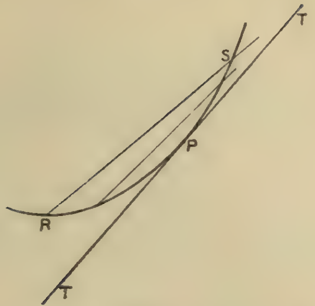


Fig. 11.

If a curve had a sharp point at P as in Fig. 12, it would not be true that the limiting position of the chord PQ is the same on whichever side of P we take Q.

In fact there are two tangents at P, viz., PT_1, PT_2, one to each branch.

Fig. 12.

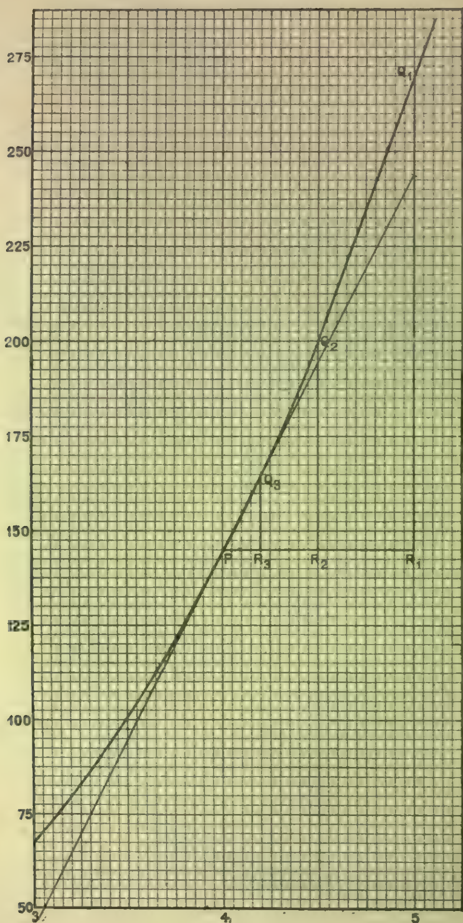


Fig. 13.

27. If one quantity be a function of another, the relation may be shewn analytically by an equation or geometrically by a graph.

e.g. in § 4 s is a function of t, this fact being expressed in the form of an equation

$$s = 5 + 3t + 2t^3.$$

This relation may be exhibited graphically.

In Fig. 13, the portion of the graph between $t = 3$ and $t = 5$ is shewn, P, Q_1, Q_2, Q_3 being the points on the graph corresponding to $t = 4$, 5, 4·5, 4·2.

The number of units of length in R_1Q_1 is the increase in s as t increases from 4 to 5 [or it is the number of feet travelled between the ends of the 4th and 5th seconds].

The increase in t is the number of units of length in PR_1.

Thus $\dfrac{\text{Increase in } s}{\text{Increase in } t}$ or what we have called the average rate of increase of s with respect to t between $t = 4$ and $t = 5$ is the gradient of the chord PQ_1.

Similarly the gradients of PQ_2 and PQ_3 give the average rates of increase of s with respect to t between $t = 4$ and $t = 4·5$ and between $t = 4$ and $t = 4·2$.

The limit to which this average rate of increase of s with respect to t tends as the increase of t is diminished more and more, or what we have called the rate of increase of s with respect to t when $t = 4$ is the limit to which the gradient of the chord PQ tends as Q is brought nearer and nearer to P, i.e. the gradient of the curve at P.

In this particular case, then, **when the space-time graph is drawn, the gradient of the curve at any point gives the speed at the instant corresponding to that point.**

Similarly **if the speed-time graph is drawn, the gradient of the curve at any point gives the acceleration at the instant corresponding to that point.**

Similarly, if we draw the graph $pV = 1200$ the gradient of PQ (Fig. 14) $\left(-\frac{3\cdot33}{5}\right)$ is the average rate of increase of p with

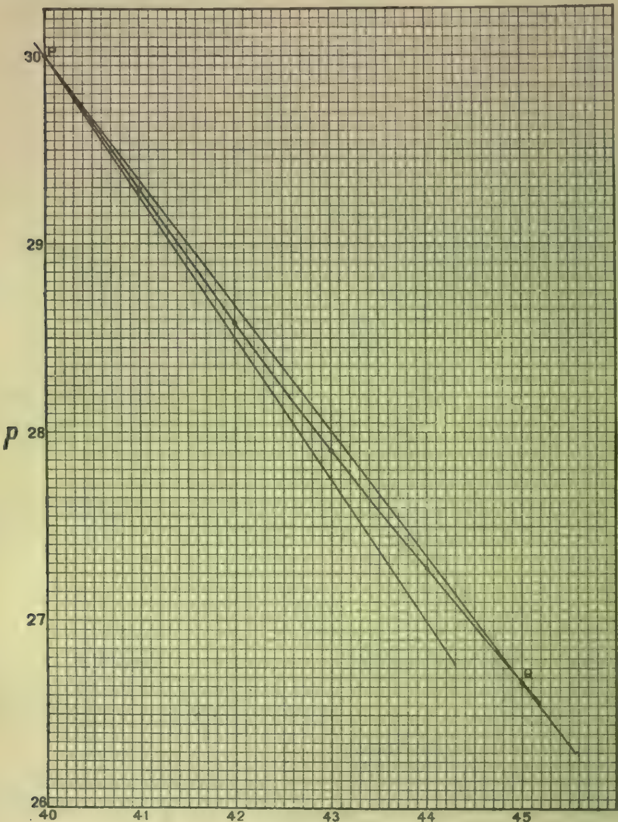


Fig. 14.

respect to V between $V = 40$ and $V = 45$. **The gradient of the curve at P is the rate of increase of p with respect to V when $V = 40$.**

28. There are cases in which the statement in § 5 appears not to be true.

e.g., suppose a marble falls vertically from a height of 36 feet. It strikes the ground with a speed of 48 ft./sec., and will rebound with reduced speed, say 40 ft./sec.

Fig. 15 is the space-time graph, the abscissae representing the number of seconds from the instant when it begins to fall and the ordinates the height above the ground in feet.

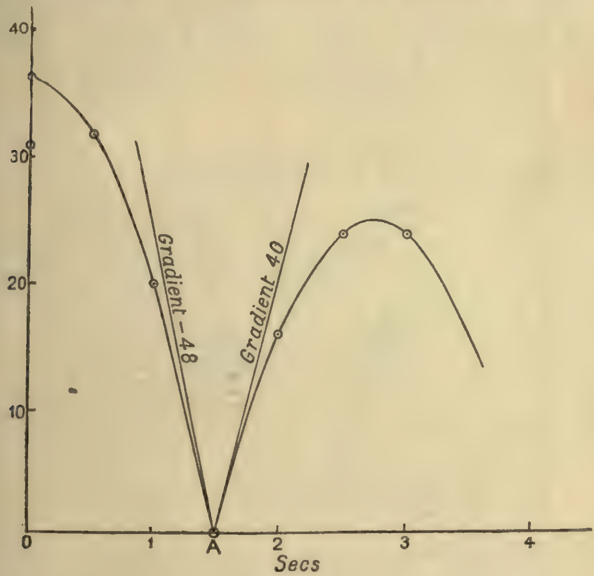


Fig. 15.

The speed 48 ft./sec. is the limit of the average speed taken over a short interval preceding the instant when it strikes the ground, and the speed 40 ft./sec. is the limit of the average speed taken over a short interval following the instant when it leaves the ground.

If we take these two instants to be the same, we seem to have two different gradients at A, or in other words to have two different speeds at the instant of impact according as we consider the limit of the average speed over a short interval preceding or over a short interval following the instant.

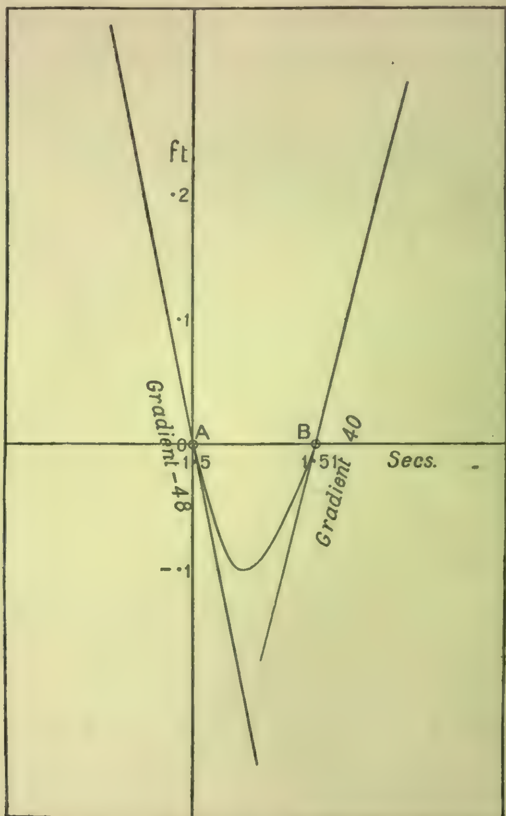


Fig. 16.

As a matter of fact the two instants are not the same and the marble is in contact with the ground for some short time during which there is compression followed by recovery.

Suppose for instance that the time of contact is $\frac{1}{100}$ sec. Fig. 16 shews roughly what the true space-time graph is like.

It is a portion of the graph, the horizontal and vertical scales being each magnified 100 times as compared with Fig. 15.

A corresponds to the instant when the marble strikes the ground and B to the instant when it leaves the ground.

It will now be seen that the statement in § 5 still applies to the speeds at the instants corresponding to A and B.

EXERCISES. VI.

1. Draw pretty accurately between $x=3$ and $x=5$ the graph of $y=\log_{10}x$.

Draw by eye the tangent at the point corresponding to $x=4$.

What is the rate of increase of $\log_{10}x$ per unit increase in x when $x=4$?

2. A body is moving in a straight line and its distances (s ft.) from a fixed point in the line at the end of (t secs.) are given by the following table

t	0	1	2	3	4	5
s	0	7	22	45	76	115

Draw as accurate a graph as possible and from it get the speed at the end of 3 secs.

3. Get the results of **Exs. IV.—5**, p. 18, graphically.

29. The processes by which we arrived at our results in the preceding illustrations were in principle the same.

In finding the speed of a moving body at a given instant, we calculated the average speed during a short interval of time starting from the instant in question. We then found that as the interval over which the average speed was calculated was made shorter and shorter, the average speed continually approached some definite speed from which we could make it

differ by as little as we pleased by making the interval short enough. This speed we called the speed of the body at the given instant.

In finding the gradient of a curve at a given point we calculated the gradient of a line passing through the given point and cutting the curve again in a neighbouring point. We then found that as the neighbouring point was moved nearer and nearer to the given point, this gradient continually approached some definite value from which we could make it differ by as little as we pleased by making the points close enough together. This value we called the gradient of the curve at the given point.

In finding the rate at which y is changing with respect to x for a given value of x, we made a small change in x, calculated the corresponding change in y and found the ratio of the change in y to the change in x. We then found that as the change in x was made smaller and smaller, this ratio continually approached some definite value from which we could make it differ by as little as we pleased by making the change in x small enough. This value we called the rate at which y was changing with respect to x for the given value of x.

30. Our final result was in each case obtained by finding the ratio of two numbers each of which was supposed to become less than any assigned number however small.

Now it might at first sight appear that if two quantities are constantly decreasing they must become more and more nearly equal. This is because we have a wrong idea of the test of approximate equality; we think the two quantities must approach to equality because their difference becomes smaller and smaller.

In estimating whether two quantities are nearly equal or not, we must look not merely at the difference between them, but at the ratio which this difference bears to the quantities compared. For example, two lengths of 3 and 4 inches are not so nearly equal as two lengths of 100 and 101 inches, for though the actual difference is the same in both cases, viz. :—1 inch, this difference

is $\frac{1}{3}$ of the smaller length in the first case, but only $\frac{1}{100}$ of the smaller length in the second case. Lengths of 1000 and 1003 inches are still nearer to equality, for though the actual difference is 3″ and therefore greater than the difference in either of the other two cases, this difference is only $\frac{3}{1000}$ of the smaller length, and $\frac{3}{1000} < \frac{1}{100}$.

The point is that there are different standards of size in the three cases. The same difference of length is of less importance when we are dealing with 1000 inches than when we are dealing with only 3 or 4.

An error of 1 foot made in measuring the length of a room would be considerable, but the same error made in finding the height of a mountain would be of very little account.

A loss of £200 would be serious for a man with an income of £300, but of little moment to a man with an income of £100,000.

An error of 10 million miles in the Sun's distance (about 90 millions of miles) would be considerable, but the same error would be inappreciable in the distance of Sirius (more than 500,000 times that of the Sun).

A carpenter would neglect an error of $\frac{1}{1000}$ of an inch, in fact he would not detect it, but in cases where it was necessary to work to $\frac{1}{10000}$ of an inch, it would be a gross error.

We have no right to say that all microscopic creatures are of the same length, because they are all so small as to be invisible to the naked eye. In comparing two such creatures, we must, so to speak, think microscopically, i.e. use a microscopic standard of length. It is just as true that a creature $\frac{1}{50}$ of a millimetre

in length is twice as long as a creature $\frac{1}{100}$ of a millimetre in length as that a man 6 ft. in height is twice as high as a child of 3 ft. in height.

To say that the difference between two quantities is *great* or *small* is meaningless. The terms are purely relative and both may be used of the same quantity in different cases. The important thing is the ratio of the difference to the quantities compared.

The test of approximate equality might be put in another way:—

$$\frac{4}{3} = 1{\cdot}3333..., \quad \frac{101}{100} = 1{\cdot}01, \qquad \frac{1003}{1000} = 1{\cdot}003,$$

and we say that the last pair of numbers are more nearly equal than either of the other pairs, because their ratio is nearer to 1 than either of the other ratios.

EXERCISES. VII.

Which of the following pairs are most and which least nearly equal?

1. (*a*) 7″ and 8″; (*b*) 9 ft. 8 ins. and 11 ft.; (*c*) ·0054″ and ·0062″.

2. (*a*) 93,000,000 miles and 92,900,000 miles.
(*b*) 9·30 inches and 9·29 inches.
(*c*) 93,000,000 miles and 93,100,000 miles.
(*d*) 930 yds. and 931 yds.

3. (*a*) x and y; (*b*) 10^8x and 10^8y; (*c*) $\frac{x}{999}$ and $\frac{y}{1000}$;
where $y = 1{\cdot}01x$.

31. We shall now give a few geometrical illustrations to shew that if x and y are two quantities which continually decrease towards zero and become zero (or vanish) together, the ratio of x to y may

(1) increase continually, i.e. it may be greater after a decrease than it was before, still greater after a further decrease and so on, becoming greater than any number that can

be named, however large, if the decrease in x and y be carried far enough;

or (2) decrease continually, i.e. it may be less after a decrease than it was before, still less after a further decrease and so on, becoming less than any fraction that can be named, however small, if the decrease in x and y be carried far enough;

or (3) increase or decrease towards a **limit**, i.e. continually approach some value which it never reaches, but to which it can be brought as near as we like if the decrease of x and y be carried far enough;

or (4) always remain the same.

32. **(1)** Fig. 17 shews a circle centre O. OA, OB are two radii at right angles. P is a point which is supposed to move along the circumference nearer and nearer to A. PM is perpendicular to OA and AP is produced to meet OB produced in Q.

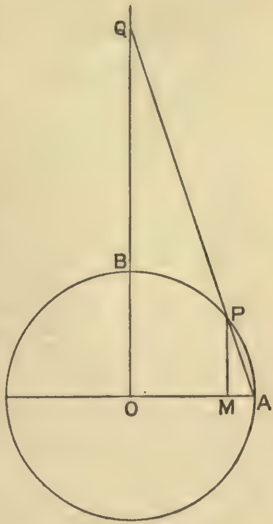


Fig. 17.

As P approaches A, PM and MA both diminish without limit, i.e. assign any length however small, we can find a position of P so that PM or MA shall be less than that length. If we suppose P to be at A, there is no triangle PMA at all, or as we may put it, PM and MA vanish together, but we have no right to say that PM and MA are all the time tending to become equal.

For every position of P which is distinct from A, PMA is similar to $\triangle$QOA, i.e. $\frac{PM}{MA} = \frac{QO}{OA}$.

But as P approaches A, Q recedes further and further from O, i.e. QO increases without limit,

and OA remains the same, $\therefore \frac{QO}{OA}$ increases without limit, i.e. $\frac{PM}{MA}$ increases without limit, i.e. state any number however great, PM may be made to contain MA more than that number of times before P reaches A.

Suppose the radius of the circle to be 1″ and call $\angle AOP\ \theta$. Then $PM = \sin\theta$ ins. and $MA = (1-\cos\theta)$ ins. We can form this table:

θ	20°	10°	5°	$2\frac{1}{2}$°	1°
PM	·3420	·1736	·0872	·0436	·01745
MA	·0603	·0152	·0038	·00095	·00015
$\frac{PM}{MA}$	5·67	11·43	22·90	45·83	114·6

θ	30′	10′	4′	2′	
PM	·0087265	·0029089	·0011636	·0005818	
MA	·0000381	·0000042	·0000007	·0000002	
$\frac{PM}{MA}$	229	688	1719	3438	etc.

[N.B. Since $\frac{\sin\theta}{1-\cos\theta} = \cot\frac{\theta}{2}$ the numbers in the last line can be written down from a table of cotangents.]

Suppose our eyes incapable of seeing any length less than $\frac{1}{1000}$th of an inch; then when $\angle AOP$ is about $2\frac{1}{2}$°, MA is invisible but PM is more than $\frac{1}{25}$ of an inch.

Again, when the angle is about 4′, PM is just visible, but we should have to use a microscope magnifying about 1700 times to make MA visible.

33. (2) If in the same figure we consider $\frac{MA}{PM}$, the ratio can be made less than any fraction we like to assign, however small, before P reaches A.

34. (3) In Fig. 18, OA is a fixed radius of a circle, centre O. P, Q are two points on the circumference, so that Q is the mid-point of the arc AP; PM, QN are perpendicular to OA.

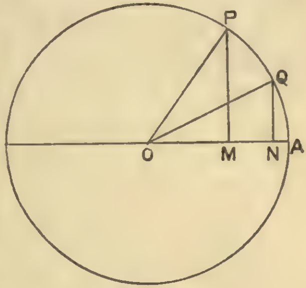


Fig. 18.

As P approaches A, Q approaches A, and AM, AN both diminish without limit. If we suppose P to have arrived at A, Q will also be at A, and AM, AN will have ceased to exist. But we must not say that AM, AN are all the time tending to equality.

Call $\angle$ AOQ θ, then $\angle$ AOP $= 2\theta$.

If the radius of the circle be r,

$$AN = r(1-\cos\theta) = 2r\sin^2\frac{\theta}{2}, \qquad AM = r(1-\cos 2\theta) = 2r\sin^2\theta = 8r\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2},$$

$$\therefore \frac{AM}{AN} = 4\cos^2\frac{\theta}{2}.$$

Now as θ diminishes $\cos\frac{\theta}{2}$ increases and can be made as near to 1 as we please if θ be made small enough.

Thus $\frac{AM}{AN}$ is a continually increasing ratio whose limit is 4, i.e. $\frac{AM}{AN}$ is always less than 4, but state a number whose defect from 4 is as little as we please, we can place P so that $\frac{AM}{AN}$ shall be equal to this number.

θ	$10°$	$5°$	$1°$	$30'$	$10'$	$2'$
$\frac{AM}{AN}$	3·9696	3·9924	3·9997	3·99992	3·999992	3·999999...

35. (4) In Fig. 19, let XOY be a triangle right-angled at X, such that

$$OX = 2XY,$$

and let a point P move along YO towards O. Draw PM perpendicular to OX.

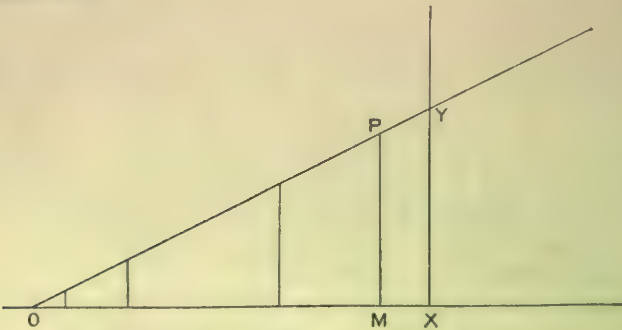


Fig. 19.

Then as P approaches O, PM and OM both diminish without limit. If we suppose P to have reached O the $\triangle$PMO has ceased to exist. But for every position of P distinct from O, however near to it

$$\frac{OM}{MP} = \frac{OX}{XY} = 2,$$

i.e. $\frac{OM}{MP}$ is a constant ratio, however small OM and MP are.

36. Of course, it may happen, that, if two quantities continually approach zero and vanish together, their ratio continually approaches the limit 1, but it must be borne in mind that this is an exceptional case.

The following illustration is an important example of this case.

In Fig. 20, if the point P move along the circumference of a circle towards A, the arc PA and the chord PA diminish together and can each be made less than any assigned length, however small,

if P be brought near enough to A; but we have no right to conclude from this that the ratio $\frac{\text{arc AP}}{\text{chord AP}}$ continually approaches 1.

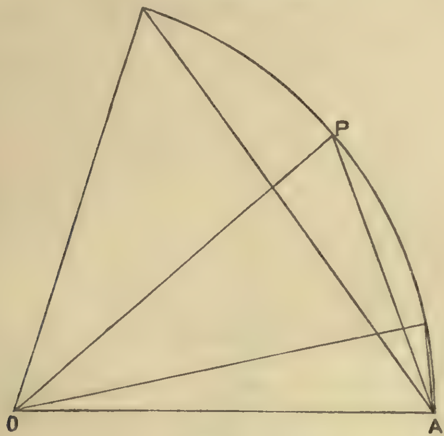


Fig. 20.

As a matter of fact it does, as may be seen from the table below.

$\angle AOP$ in degrees	10°	9°	8°	7°	6°
Arc AP	·1745329	·1570796	·1396263	·1221730	·1047198
Chord AP	·1743114	·1569182	·1395130	·1220970	·1046720
$\frac{\text{Arc } AP}{\text{Chord } AP}$	1·0013	1·00103	1·0008	1·0006	1·0005

$\angle AOP$ in degrees	5°	4°	3°	2°	1°
Arc AP	·0872665	·0698132	·0523599	·0349066	·0174533
Chord AP	·0872388	·0697990	·0523538	·0349048	·0174530
$\frac{\text{Arc } AP}{\text{Chord } AP}$	1·0003	1·0002	1·0001	1·00005	1·00002

37. Another difficulty which often occurs is this :—

How can a quantity continually decrease without eventually disappearing altogether?

In Fig. 21, OAB is a quadrant of a circle, centre O.

BX is the tangent at B.

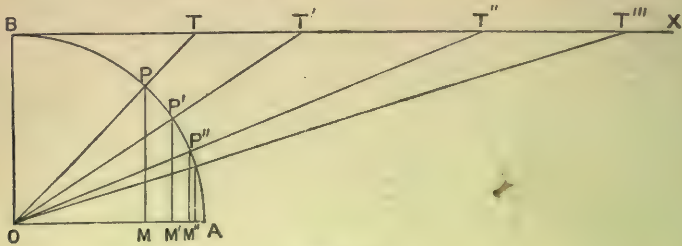


Fig. 21.

Suppose T to travel along BX, at any rate whatever, say 1000 miles an hour, and in every position of T, join OT cutting the circumference in P. It is obvious that as T travels along BX, P moves nearer to A, but P will never reach A, however far T be supposed to travel along BX.

The lengths PM, MA considered in § 32 can be supposed to go on decreasing for ever without actually disappearing and it has been shewn that during the motion of P towards A, $\frac{PM}{MA}$ will increase without limit.

CHAPTER II

DIFFERENTIATION FROM FIRST PRINCIPLES

38. We shall now go through some of the investigations of Chapter I in a slightly different manner, using the notation of the subject.

39. We have

$$s = 5 + 3t + 2t^3,$$

giving the distance s feet travelled in t seconds.

If we take a slightly greater value of t, say $\mathbf{t + \Delta t}$, we shall obtain a slightly different value of s, say $\mathbf{s + \Delta s}$.

[Δt is merely shorthand for "an 'increment' or 'increase' in the value of t," Δ being the Greek equivalent of capital D. Δ by itself means nothing, and Δt must be considered as one symbol; in fact, Δt takes the place of h in § 4.]

There will be the same relation between $(s + \Delta s)$ and $(t + \Delta t)$ as between s and t;

i.e.

$$\begin{aligned} s + \Delta s &= 5 + 3\,(t + \Delta t) + 2\,(t + \Delta t)^3 \\ &= 5 + 3t + 2t^3 + (3 + 6t^2)\,\Delta t + 6t\,(\Delta t)^2 + 2\,(\Delta t)^3, \end{aligned}$$

$$\therefore\ \Delta s = (3 + 6t^2)\,\Delta t + 6t\,(\Delta t)^2 + 2\,(\Delta t)^3.$$

This gives the extra distance travelled in the extra interval of time Δt.

[e.g. If we want the distance travelled in the half-second following the end of the 3rd second, we put $t=3$, $\Delta t=\frac{1}{2}$.

$$\therefore \Delta s = 57\times\frac{1}{2}+18\times\frac{1}{4}+2\times\frac{1}{8}=33\tfrac{1}{4}],$$

$$\therefore \frac{\Delta s}{\Delta t}=(3+6t^2)+6t\,.\,\Delta t+2\,(\Delta t)^2.$$

Since Δs feet is the distance described in the interval Δt secs

$\therefore \frac{\Delta s}{\Delta t}$ gives the average speed during this interval in ft./sec.

As Δt is made smaller and smaller $6t\,.\,\Delta t+2\,(\Delta t)^2$ becomes smaller and smaller and can be made as small as we please if Δt is made small enough.

i.e. $\frac{\Delta s}{\Delta t}$ can be made as near to $(3+6t^2)$ as we please if Δt be made small enough.

The limit to which $\frac{\Delta s}{\Delta t}$ continually approaches and from which we can make it differ by as little as we please by taking Δt small enough is called $\frac{ds}{dt}$.

Thus $\frac{ds}{dt}=3+6t^2$ and $\frac{ds}{dt}$ ft./sec. is the speed of the body at the end of t seconds.

[*V.* definition of speed at an instant § 4.]

40. Notice that if $s=5+3t+2t^3$

(i) $\frac{\Delta s}{\Delta t}=(3+6t^2)+6t\,.\,\Delta t+2\,(\Delta t)^2$ is an accurate statement whatever the value of Δt, e.g. take $t=4$, $\Delta t=\frac{1}{2}$ and we get $99+12+\frac{1}{2}$ or $111\frac{1}{2}$ ft./sec. as the average speed during the interval of $\frac{1}{2}$ second immediately after the end of the 4th second.

(ii) $\frac{\Delta s}{\Delta t} = 3 + 6t^2$ is approximately true if Δt be small and becomes more and more nearly true as Δt is made smaller, but there is no value of Δt however small for which the statement is accurately true.

[See table in § 4 where the third column gives $\frac{\Delta s}{\Delta t}$ for different values of Δt when $t = 4$ and in this case $3 + 6t^2 = 99$.]

In other words, the average speed during a small interval Δt following the end of t seconds is approximately $(3 + 6t^2)$ ft./sec. and this is more nearly true, the shorter we make the interval Δt.

e.g. if we want approximately the distance described in $\frac{1}{100}$th of a second after the end of the 3rd second we have

$$\Delta s = (3 + 6t^2)\,\Delta t$$

approximately, meaning that if Δt be small the distance described in Δt seconds immediately following the end of t seconds is nearly $(3 + 6t^2)\,\Delta t$ feet.

Put $t = 3$ and $\Delta t = \frac{1}{100}$ and we have approximately as the distance required in our special case, $\frac{57}{100}$ or ·57 ft.

[Shew that the distance is really ·571802 ft.]

This gives an error of about ·3 °/。.

If we used the formula to give the distance in $\frac{1}{1000}$ sec. we should get ·057 ft. instead of ·057018002 ft., an error of about ·03 °/。 and so on.

(iii) $\frac{ds}{dt} = 3 + 6t^2$ is accurately true. It is in fact a convenient way of saying that the unattainable limit which $\frac{\Delta s}{\Delta t}$ is trying to reach is $3 + 6t^2$, or $\frac{ds}{dt}$ stands for the ideal value which $\frac{\Delta s}{\Delta t}$ is striving to attain.

41. *ds* and *dt* have no separate meanings. *dt* does not stand for a very small increment in *t*, nor *ds* for a small increment in *s*, for we have just seen that however small such increments are made, their ratio is never $3 + 6t^2$.

$\frac{ds}{dt}$ then must be treated as a whole, and simply means the limit to which $\frac{\Delta s}{\Delta t}$ is constantly tending as Δt and with it Δs becomes smaller and smaller. $\frac{\Delta s}{\Delta t}$ can be brought as near as we please to $\frac{ds}{dt}$ if we make Δt small enough.

$\frac{ds}{dt}$ does not mean the result of dividing ds by dt as $\frac{\Delta s}{\Delta t}$ means the result of dividing Δs by Δt. The fractional form $\frac{ds}{dt}$ is retained merely to remind us of the fraction of which it is the limiting value.

42. It is a common error to say that $\frac{ds}{dt}$ is the value of $\frac{\Delta s}{\Delta t}$ when Δs and Δt are each zero. If this statement be examined, it will be seen to have no meaning.

Let us go back to the stage at which $\frac{\Delta s}{\Delta t}$ was obtained.

We had $\Delta s = (3 + 6t^2)\,\Delta t + 6t\,.\,(\Delta t)^2 + 2\,.\,(\Delta t)^3.$

We then divided both sides of the equation by Δt.

Now if $\Delta t = 0$ we have no right to do this.

[It is certainly true that $8 \times 0 = 3 \times 0$ but we cannot divide both sides by 0 and deduce $8 = 3$. Similarly if $x \times 0 = y \times 0$ we cannot conclude that $x = y$, though if $x \times \frac{1}{10^{20}} = y \times \frac{1}{10^{20}}$ it is true that $x = y$.]

We should in fact be trying to estimate the speed of the body from the fact that it had travelled 0 feet in 0 secs.

43. We have found that the speed at the end of t seconds is $(3+6t^2)$ ft./sec.

The result obtained in § 4 is merely a particular case of this, for when $t=4$, $3+6t^2=99$ and our formula enables us to write down the speed at any instant, e.g. to get the speed at the end of 10 seconds, put $t=10$ and we get 603 ft./sec.

EXERCISES. VIII.

1. Obtain a formula giving the speed at the end of t seconds of a body whose distance (s feet) from a fixed point at the end of t seconds is given by

(i) $s=16t^2$, (ii) $s=100t-16t^2$,

(iii) $s=5+t^3$, (iv) $s=5t+9$,

and in each case find the speeds at the end of 3, 5, 17 seconds.

2. If $s=16t^2$, what is the speed at the end of 2 seconds? Assuming this speed to remain constant for the next tenth of a second, how far would the body move, and what percentage error would be made in taking this as the true distance? Do the same for a hundredth of a second.

3. If the speed of a body increase by 12 ft./sec. in 3 seconds, what is the average acceleration during the 3 seconds?

4. If the speed at the end of 4 seconds is 78 ft./sec. and at the end of 9 seconds 146 ft./sec., what is the average acceleration between the end of the 4th and the end of the 9th second? [v. Ex. 7, p. 19.]

5. If the speed at the end of t seconds is v ft./sec. and at the end of $(t+\Delta t)$ seconds, $(v+\Delta v)$ ft./sec., what is the average acceleration during the interval Δt? By what symbol should the acceleration at the end of t seconds be denoted?

6. In Ex. 1, find the accelerations at the end of 3, 5, 17 seconds.

7. With the formula $s=5+t^3$ what is approximately the distance described in $\frac{1}{50}$ of a second immediately following the end of the 3rd second?

What is approximately the increase of speed during this interval?

44. Definition. $\frac{ds}{dt}$ is called the **differential coefficient** of s with respect to t, and "to differentiate s with respect to t" means to find $\frac{ds}{dt}$.

45. Instead of saying that if $s = 3t + 2t^3$ then $\frac{ds}{dt} = 3 + 6t^2$ we might say $\frac{d(3t+2t^3)}{dt} = 3 + 6t^2$. This form of stating the result will sometimes be found convenient.

46. Consider a square plate whose side is x ins. and let its area be y sq. ins. so that $y = x^2$.

Suppose the side increased to $(x + \Delta x)$ ins., and as a result let the area be increased to $(y + \Delta y)$ sq. ins.

$$\therefore \; y + \Delta y = (x + \Delta x)^2,$$

$$\therefore \; \Delta y = 2x\Delta x + (\Delta x)^2.$$

i.e. the increase in area due to an increase Δx in the side is $2x\Delta x + (\Delta x)^2$ sq. ins.

[In Fig. 22 each of the shaded rectangles has area $x\Delta x$ and the square in the top corner has area $(\Delta x)^2$.]

$$\therefore \; \frac{\text{Increase in number of sq. ins. in area}}{\text{Increase in number of ins. in side}} = \frac{\Delta y}{\Delta x} = 2x + \Delta x$$

and as before we deduce $\frac{dy}{dx} = 2x$.

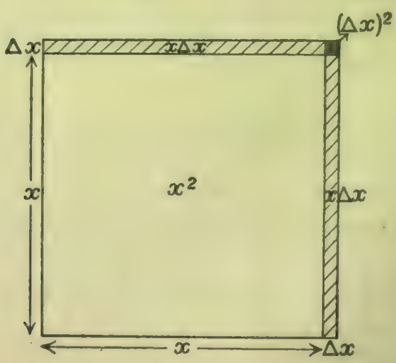


Fig. 22

47. As before notice that if $y = x^2$

(i) $\dfrac{\Delta y}{\Delta x} = 2x + \Delta x$ is an accurate statement whatever the value of Δx, e.g. if $x = 3$, and $\Delta x = \cdot 2$,

$$\frac{\Delta y}{\Delta x} = 6\cdot 2, \quad \text{i.e. Increase in } y = 6\cdot 2 \text{ times increase in } x$$
$$= 6\cdot 2 \times \cdot 2 = 1\cdot 24.$$

i.e. if the side be originally 3″ and increase by ·2″ the area increases by 1·24 sq. in.

(ii) $\dfrac{\Delta y}{\Delta x} = 2x$ is approximately true and becomes more nearly true as Δx is made smaller.

e.g. if $x = 3$, $\Delta y = 6\Delta x$ nearly, i.e. if a square have a side of 3 ins. and a small increase be made in the side, the increase in the number of sq. ins. in the area will be nearly 6 times the increase in the number of ins. in the side.

Suppose for example the side increases from 3 to 3·02 ins. the area increases approximately by ·12 sq. ins.

Actually the increase is ·1204 sq. in. [·0004 sq. in. is the area of the small square in Fig. 22.]

(iii) $\dfrac{dy}{dx} = 2x$ is accurately true, and is simply a convenient way of saying that the approximate statement in (ii),

$$\frac{\text{Increase in } y}{\text{Increase in } x} = 2x,$$

can be brought as near to the truth as we please by making the increase in x small enough. Taking again the special case when $x = 3$, $\dfrac{dy}{dx} = 6$, this might be stated thus:

When $x = 3$, the rate at which y is increasing compared with x is 6, or the x-rate of increase of y is 6, or the rate of increase of y per unit increase of x is 6.

48. If we introduce the idea of time into the question and suppose the side of the square to be steadily growing, then in the same time in which Δx ins. is added to the side, Δy sq. ins. is added to the area and $\frac{\Delta y}{\Delta x} = 6 \cdot 2$ tells us that y increases $6 \cdot 2$ times as much as x increases in the same time. If we call the time in which the change takes place Δt secs. we have

$$\frac{\Delta y}{\Delta t} \Big/ \frac{\Delta x}{\Delta t} = 6 \cdot 2,$$

i.e. average rate of increase of y during this interval $= 6 \cdot 2$ times the average rate of increase of x during the interval.

In the general case

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta t} \Big/ \frac{\Delta x}{\Delta t} = 2x + \Delta x.$$

If Δt, and therefore at the same time Δx and Δy, be made smaller and smaller, $\frac{\Delta y}{\Delta t}$ and $\frac{\Delta x}{\Delta t}$ can be made as near as we please to $\frac{dx}{dt}$ and $\frac{dy}{dt}$ respectively, $\frac{\Delta y}{\Delta x}$ as near as we please to $\frac{dy}{dx}$, and $2x + \Delta x$ as near as we please to $2x$.

$\therefore$ we get
$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = 2x.$$

But $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are what we call the rates at which y and x respectively are increasing at the end of t secs.

Thus $\frac{dy}{dx} = 2x$ being equivalent to $\frac{dy}{dt} = 2x \cdot \frac{dx}{dt}$ may be looked upon as telling us that y is increasing $2x$ times as fast as x.

In the special case when $x = 3$, $\frac{dy}{dx} = 6$, and this tells us that at the particular instant when the side of the square is 3 inches y is increasing 6 times as fast as x, or the number of square inches in the area is increasing 6 times as fast as the number of inches in the side.

49. *Example.* The side of a square increases uniformly at the rate of ·1 inch per second. At what rate is the area growing when the side of the square is 6 inches?

If y sq. ins. is the area of a square of side x ins.

$$y = x^2,$$

$$\therefore \frac{dy}{dx} = 2x,$$

and when $x = 6$, $$\frac{dy}{dx} = 12.$$

i.e. at the instant when the side of the square is 6 ins. y is increasing 12 times as fast as x, or the number of square inches in the area is increasing 12 times as fast as the number of inches in the side.

But x is increasing ·1 per sec.

$\therefore$ y „ „ $12 \times \cdot 1$ or $1{\cdot}2$ per sec.

i.e. the area is increasing at the rate of **1·2** sq. in. per sec.

50. Suppose the pressure (p lbs./sq. in.) and the volume (V cub. ins.) of a given mass of gas are connected by the relation

$$pV = 1200 \text{ or } p = \frac{1200}{V}.$$

(1) If we take a slightly different value of V, $V + \Delta V$ we shall get a slightly different value of p, $p + \Delta p$ such that

$$p + \Delta p = \frac{1200}{V + \Delta V},$$

$$\therefore \Delta p = \frac{1200}{V + \Delta V} - \frac{1200}{V} = -\frac{1200\Delta V}{(V + \Delta V)V}.$$

(2) $$\therefore \frac{\Delta p}{\Delta V} = -\frac{1200}{(V + \Delta V)V}.$$

As ΔV is made smaller and smaller this comes nearer and nearer to $-\frac{1200}{V^2}$ and can be made as near to $-\frac{1200}{V^2}$ as we like if ΔV be made small enough.

(3) $$\therefore \frac{dp}{dV} = -\frac{1200}{V^2}.$$

51. The meaning of the negative sign is simply this :—

As V increases p decreases, so that ΔV and Δp have opposite signs, therefore $\frac{\Delta p}{\Delta V}$ is negative.

We might read the final result either

The rate of increase of p per unit increase of $V = -\frac{1200}{V^2}$,

or the rate of decrease of p per unit increase of $V = +\frac{1200}{V^2}$,

or p is decreasing $\frac{1200}{V^2}$ times as fast as V is increasing.

52. As before, notice, if $pV = 1200$

(i) $\frac{\Delta p}{\Delta V} = -\frac{1200}{(V+\Delta V)V}$ is an accurate statement whatever the values of V and ΔV.

e.g. if $V = 40$, and $\Delta V = 5$

$$\frac{\Delta p}{\Delta V} = -\frac{1200}{40 \times 45} = -\frac{2}{3}.$$

i.e. decrease in $p = \frac{2}{3}$ (increase in V) $= 3\frac{1}{3}$.

(ii) $\frac{\Delta p}{\Delta V} = -\frac{1200}{V^2}$ is approximately true and becomes more nearly true the smaller we make ΔV.

e.g. if $V = 20$, $\Delta p = -3 \,.\, \Delta V$ nearly.

i.e. if the volume be 20 cub. ins. and a small increase be made in the volume, the decrease in the number of lbs./sq. in. in the pressure will be nearly 3 times the increase in the number of cub. ins. in the volume.

Suppose for example the volume increases from 20 to 20·1 cub. ft., the pressure decreases approximately ·3 lbs./sq. in., i.e. from 60 to 59·7 lbs./sq. in.

[Actually the decrease is ·2985.]

(iii) $\dfrac{dp}{dV}=-\dfrac{1200}{V^2}$ is accurately true and is merely a convenient way of saying that $\dfrac{\Delta p}{\Delta V}$ can be brought as near as we please to $-\dfrac{1200}{V^2}$ if ΔV be made small enough.

53.
$$\frac{dp}{dV}=-\frac{1200}{V^2}$$

is a formula giving the rate at which p is changing with respect to V for any assigned value of V.

$$\text{e.g. if } V=10,\ \frac{dp}{dV}=-12.$$

i.e. when the volume is 10 cubic feet,

$$p \text{ is } \left.\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix}\right\} \text{ 12 times as fast as V is } \left\{\begin{matrix}\text{decreasing}\\ \text{increasing}\end{matrix}\right..$$

EXERCISES. IX.

1. A sq. ins. is the area of a circle of radius r ins.

What is the relation between A and r?

If r be increased to $r+\Delta r$, find the corresponding increase ΔA in A, and prove as above that $\dfrac{dA}{dr}=2\pi r$.

This tells us that approximately $\Delta A=2\pi r\,.\,\Delta r$.

Interpret this as a rule for the approximate area of a thin circular ring, and apply it to find approximately the area of a circular ring whose inner radius is 3 ft. and thickness ·1 inch.

2. Shew that the percentage error made in taking $2\pi r\,.\,\Delta r$ for the area of the ring is less than $\dfrac{50\,.\,\Delta r}{r}$.

What is approximately your percentage error in Ex. 1?

3. The radius of a circle is increasing at the rate of 1 mm. per second. At what rate is the area increasing when the radius is (i) 1 mm., (ii) 1 cm., (iii) 20 cms.?

4. What will be the approximate increase in area during the next tenth of a second after the radius reaches 20 cms.?

5. The side of a square is measured and found to be 8 inches. If an error of ·01 inch is made in measuring the side, find approximately the error in the calculated area.

6. The area of a circle grows at the rate of 2 sq. ins. per second. At what rate is the radius growing

(i) when the radius is 5 inches,

(ii) when the area is 20 sq. ins.?

7. If $pv=80$, find the rate of increase of p with respect to v when $v=$ (i) 10, (ii) 80.

8. OX, OY are two lines at right angles. A and B are points in OX, OY respectively such that the area of the triangle OAB is always 10 square inches. If A moves along OX at a constant speed of 0·1 inch per second, what is the speed of B (i) when OA $=8''$, (ii) when OB $=8''$?

54. Let P be a point whose coordinates are (x, y) on the curve $y=x^2$ and let Q be a point on the curve near to P. Its coordinates will differ slightly from those of P and may be denoted by $(x+\Delta x,\ y+\Delta y)$. (Fig. 23.)

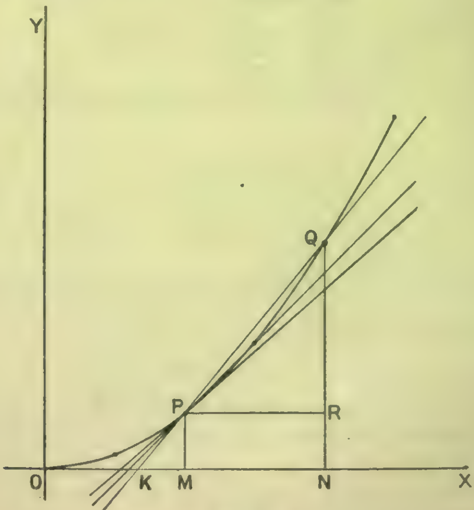


Fig. 23.

Then PR $= \Delta x$, RQ $= \Delta y$.

$$\therefore \left.\begin{array}{l}\tan \text{RPQ or } \tan \text{XKP} \\ \text{or gradient of PQ}\end{array}\right\} = \frac{\Delta y}{\Delta x} \text{ [v. note on Scale, § 25].}$$

Now
$$y + \Delta y = (x + \Delta x)^2$$
$$= x^2 + 2x\Delta x + (\Delta x)^2,$$
$$\therefore \ \mathbf{\Delta y = 2x \,.\, \Delta x + (\Delta x)^2},$$
$$\therefore \ \frac{\Delta y}{\Delta x} = 2x + \Delta x\,;$$

i.e. gradient of PQ $= 2x + \Delta x$.

Now suppose Q to move along the curve towards P, so that Δx and Δy become smaller and smaller, then by bringing Q near enough to P, we can make $\dfrac{\Delta y}{\Delta x}$ as near to $2x$ as we please.

The limit which $\dfrac{\Delta y}{\Delta x}$ continually approaches and from which it can be made to differ by as little as we please if we make Δx small enough is called $\dfrac{dy}{dx}$.

Thus in this case $\dfrac{dy}{dx} = 2x$, and this is called the gradient of the curve at the point P.

55. Notice, if $y = x^2$

(i) $\dfrac{\Delta y}{\Delta x} = 2x + \Delta x$ is an accurate statement whatever the value of Δx, e.g. take $x = 1$, $\Delta x = \cdot 2$, and we have $2{\cdot}2$ as the gradient of the chord joining the point for which $x = 1$ to the point for which $x = 1{\cdot}2$.

(ii) $\dfrac{\Delta y}{\Delta x} = 2x$ is approximately true if Δx is small and becomes more and more nearly true as Δx is made smaller, but there is no value of Δx however small for which $\dfrac{\Delta y}{\Delta x} = 2x$, in other words if

we draw a chord through P cutting the curve in a point Q as near to P as we please, the gradient of this chord will never be exactly $2x$, but can be made as near to $2x$ as we please if we bring Q near enough to P. For instance if $x=3$, $\frac{\Delta y}{\Delta x}=6$ nearly, i.e. the gradient of a chord joining (3, 9) to a neighbouring point on the curve is nearly 6 and becomes nearer to 6 the closer this neighbouring point is to (3, 9).

(iii) $\frac{dy}{dx}=2x$ is accurately true and is merely a convenient way of saying that the unattainable limit which $\frac{\Delta y}{\Delta x}$ is trying to reach is $2x$. In fact $2x$ is the gradient of a line through P towards which the chord PQ is continually tending as Q approaches P, but with which it can never be made to coincide however near Q is to P.

This line towards which PQ continually tends is called the tangent at P.

Thus $\frac{dy}{dx}$ gives the gradient of the tangent to the curve at P, or as it is sometimes called, **the gradient of the curve at P.**

56. The result $\frac{dy}{dx}=2x$ may be looked upon as a formula giving the gradient at any point of the curve. Thus, if $x=3$, $\frac{dy}{dx}=6$, i.e. the gradient of the curve at the point (3, 9) is 6.

57. Notice that although $\frac{dy}{dx}$ is not the gradient of any chord through P however near the other end of the chord may be to P, yet **it is the gradient of an actual line through P**, namely the line which we call the tangent to the curve at P.

58. We can now see better the significance of the approximation in (ii).

Let P be the point (3, 9) and let Q be a point on the curve near to P. (Fig. 24.)

Let PT the tangent at P cut QR in Q′.

Then the statement is that the gradient of PQ is nearly 6, in other words $\frac{RQ}{RP}$ is nearly 6; but since $\frac{RQ'}{RP} = 6$, this is equivalent to saying that RQ is nearly the same as RQ′.

e.g. suppose PR = ·02 then Q is the point (3·02, 9·1204).

∴ RQ = ·1204 and RQ′ = 6 × ·02 = ·12.

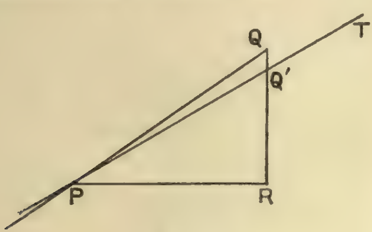


Fig. 24.

The error in taking RQ = RQ′ is about $\frac{1}{3}$ °/ₒ.

If PR = ·001 then Q is the point (3·001, 9·006001).

∴ RQ = ·006001 and RQ′ = 6 × ·001 = ·006.

The error is about $\frac{1}{60}$ °/ₒ.

59. A caution similar to that issued in § 41 applies here also. *dy* and *dx* have no separate meanings: they do not stand for small changes in length, for we have seen that however small the increments in x and y are supposed to be, their ratio is never exactly $2x$.

We must not say that $\frac{dy}{dx}$ is the value of $\frac{\Delta y}{\Delta x}$ when Δx and Δy are each zero. The whole process by which we obtained $\frac{\Delta y}{\Delta x}$

becomes unintelligible if we imagine affairs pushed so to speak to the limit. The whole argument depends on the existence of a triangle PQR, which even though every side is out of the range of visibility can be supposed magnified so as to become visible, e.g. [with unit 1″ along each axis] if $x = 1$ and $\Delta x = \cdot 0001$ then $\Delta y = \cdot 00020001$ and if we used a microscope magnifying 1000 times the triangle PQR would appear as a triangle in which $PR = \cdot 1''$ and $RQ = \cdot 20001''$. But if we push Q to absolute coincidence with P, no amount of magnifying will separate the points P, Q, R.

60. If the equation of a curve is given as $y = x^2$, we may look upon this as a formula for finding the value of y corresponding to any assigned value of x, or for finding the ordinate corresponding to any given abscissa. In other words it is a formula by means of which we may obtain the positions of any number of points on the curve.

$\dfrac{dy}{dx} = 2x$ is a formula giving the gradient corresponding to any given abscissa.

e.g. if $x = 5$, $y = 25$ and $\dfrac{dy}{dx} = 10$.

i.e. (5, 25) is the point on the curve whose abscissa is 5 and the gradient at this point is 10.

61. In drawing the curve $y = x^2$ it is a great help to make use of both formulae.

Suppose we draw it between $x = -3$ and $x = 3$, we have the following table

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9
$\dfrac{dy}{dx}$	-6	-4	-2	0	2	4	6

Plot the points

$(-3, 9)$ $(-2, 4)$ $(-1, 1)$ $(0, 0)$ $(1, 1)$ $(2, 4)$ $(3, 9)$.

Through these points draw lines whose gradients are -6, -4, -2, 0, 2, 4, 6 respectively. These lines then are to be tangents to the curve (Fig. 25).

If we try to draw a curve passing through these 7 points and touching these 7 lines, we shall find that the run of the curve is pretty well determined.

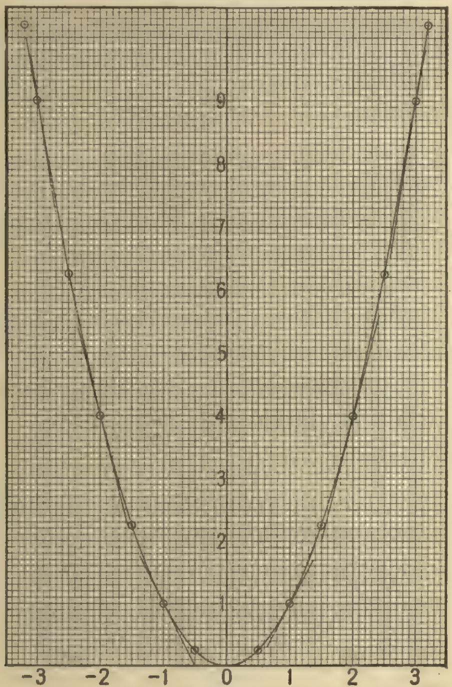


Fig. 25.

EXERCISES. X.

1. Use the result just obtained to get the gradient of $y=x^2$ at the points where

$$x=-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3.$$

Draw the graph $y=x^2$, and through the points where

$$x=-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3$$

draw lines with gradients just found. They should be tangents to your curve. Take 1″ as the x-unit and $\frac{1''}{2}$ as the y-unit.

2. Find the gradient of $y=x^3$ at the points where

$$x=-3,\ -1,\ 0,\ 5.$$

Shew that the gradient of $y=x^3$ is never negative. Draw a figure.

3. Find the gradient of $y=x^4-4x$ at the points where $x=0,\ 1,\ \sqrt[3]{4},\ 2.$ Draw a figure.

4. If $y=x^2$, what is $\frac{dy}{dx}$ when $x=7$? Suppose we move along the curve from the point where $x=7$ to the point where $x=7{\cdot}03$, what is the gradient of the chord joining these points? What would be the percentage error if we took the value of $\frac{dy}{dx}$ just obtained as the gradient?

5. Shew that the percentage error when $2x$ is taken instead of $2x+\Delta x$ or $2x\,.\,\Delta x$ instead of $2x\,.\,\Delta x+(\Delta x)^2$ is less than $\frac{50\,.\,\Delta x}{x}$ and if $x^2+2x\,\Delta x$ be taken instead of $(x+\Delta x)^2$ the percentage error $< 100\left(\frac{\Delta x}{x}\right)^2$.

Use these results to find approximately the p.c. error

(i) if $2{\cdot}01^2$ be taken as $4{\cdot}04$;

(ii) if $7{\cdot}03^2$ „ „ „ $49{\cdot}42$;

(iii) if the gradient of the chord joining the point where $x=5$ to the point where $x=5{\cdot}02$ be taken as 10;

(iv) if RQ in Fig. 24 be taken as equal to RQ′, P being the point (10, 100) and PR $={\cdot}5$.

6. Find (i) $\frac{d\,(3x-x^2)}{dx}$, (ii) $\frac{d\,(8t)}{dt}$, (iii) $\frac{d\left(\frac{1}{v^2}\right)}{dv}$.

62. The process by which we obtained $\frac{ds}{dt}$ in § 39, $\frac{dy}{dx}$ in §§ 46 and 54, $\frac{dp}{d\mathrm{V}}$ in § 50 should be carefully studied, as it is the same in all cases.

Taking the first case:—We had s given as a function of t $[s = 5 + 3t + 2t^2]$.

(1) We gave t a small increment (Δt) and found the corresponding increment in s (Δs).

(2) We found the ratio $\frac{\Delta s}{\Delta t}$.

(3) We found the limit to which $\frac{\Delta s}{\Delta t}$ tended as Δt was made smaller and smaller, and from which it could be made to differ by as little as we please by taking Δt small enough and this limit was $\frac{ds}{dt}$.

63. Generally if a quantity y be a function of another quantity x, a small change, Δx, in x, will produce in y a corresponding small change which we call Δy, and in the cases with which we have to deal $\frac{\Delta y}{\Delta x}$ tends continually to some limit from which it can be made to differ by as little as we please by making Δx small enough. This limit is denoted by $\frac{dy}{dx}$, and in the Differential Calculus the fundamental problem is to determine $\frac{dy}{dx}$ for different forms of the relation connecting y and x.

64. $\frac{dy}{dx}$ gives the rate of increase of y per unit increase of x or the rate of increase of y with respect to x for any value of x. It tells us that y is increasing so many times as fast as x.

In the special case when y and x are replaced by s and t with the usual meanings, $\frac{ds}{dt}$ gives the speed at a given instant.

Similarly $\frac{dv}{dt}$ gives the acceleration at a given instant.

If y and x are the coordinates of a point on a curve whose equation is the given equation connecting y and x, $\frac{dy}{dx}$ gives the gradient of the curve at any point.

65. If we draw the graph of

$$s = 5 + 3t + 2t^3$$

[Fig. 26 shews it drawn between $t = 0$ and $t = 3$.]

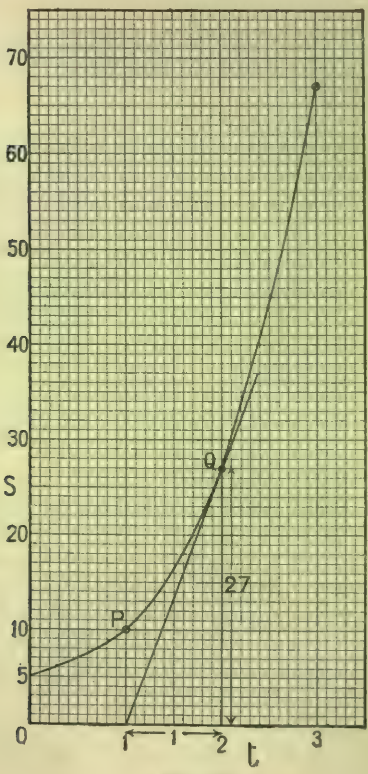


Fig. 26.

the gradient at any point corresponding to time t will be $\frac{ds}{dt}$ or $3 + 6t^2$.

But $\frac{ds}{dt}$ gives the speed at the end of time t.

Thus we see that if the space-time graph be drawn the speed at any instant is given by the gradient of the graph at the point corresponding to that instant. (See § 27.)

e.g. in the figure Q is the point corresponding to $t = 2$, $s = 27$.

The gradient of the tangent at Q is 27. The speed at the end of 2 seconds is 27 ft./sec.

66. Similarly if we draw the speed-time graph

$$v = 3 + 6t^2,$$

the gradient at any point is $\frac{dv}{dt}$ [shew that this is $12t$] and $\frac{dv}{dt}$ gives the acceleration in ft./sec.^{2}

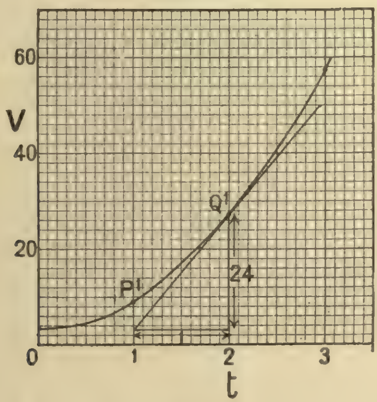


Fig. 27.

Thus if the speed-time graph be drawn the acceleration at any instant is given by the gradient of the graph at the point corresponding to that instant.

e.g. Q′ is the point corresponding to $t=2$, $v=27$.

The gradient of the tangent is 24. The acceleration at the end of 2 secs. is 24 ft./sec.^{2}

67. Generally if y is given as a function of x [or $y=f(x)$, $f(x)$ being algebraic shorthand for a function of x] and if the graph of $y=f(x)$ be drawn, the rate at which y is increasing compared with x for any value of x is the gradient of the graph at the point corresponding to this value of x.

Thus if we could draw accurately the graph of $y=f(x)$ and also draw accurately the tangent at a given point we should be able to read off the value of $\frac{dy}{dx}$ for the value of x corresponding to that point.

EXERCISES. XI.

[To be done as accurately as possible by drawing tangents; v. § 67.]

1. Draw the graph of $y=\sqrt{25-x^2}$ or $x^2+y^2=25$.

It is a circle centre at the origin, radius 5.

What is $\frac{dy}{dx}$ when $x=1, 3, -4$?

2. Draw the graph of $y=\frac{1}{x}$ between $x=0$ and $x=5$.

What is $\frac{dy}{dx}$ when $x=1, 2$?

68. *Example.* Find from first principles $\frac{dy}{dx}$ when $y=x^3$.

Let x receive a small increment Δx and let the resulting increment in y be Δy.

Then $y + \Delta y = (x + \Delta x)^3$

$$= x^3 + 3x^2 . \Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

and $$y = x^3,$$

$$\therefore\ \Delta y = 3x^2 . \Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$$

$$\therefore\ \frac{\Delta y}{\Delta x} = 3x^2 + 3x . \Delta x + (\Delta x)^2.$$

Now if Δx be made smaller and smaller, each of the last two terms on the right can be made as small as we please by taking Δx small enough.

$\therefore\ \frac{\Delta y}{\Delta x}$ can be made as near to $3x^2$ as we please by making Δx small enough.

$$\therefore\ \frac{\mathrm{dy}}{\mathrm{dx}} = 3\mathrm{x}^2.$$

EXERCISES. XII.

1. Find $\frac{dy}{dx}$ if $y = k . x^3$ where k is any constant.

Make use of the result of Ex. 1 to solve the remaining questions.

2. If a body is travelling in a straight line and its distance from a fixed point in the line at the end of t seconds is given by $s = 4t^3$, where s is the number of feet, find the speed at the end of 3 seconds.

3. If the graph $y = 5x^3$ be drawn, what is the gradient at the point (2, 40) and what is the equation of the tangent at that point?

4. If the edge of a cube be increasing at the rate of ·001 of an inch per second, find at what rate the volume is increasing when the edge is 2 feet.

5. If the radius of a sphere be increasing at the rate of 3 inches per second, at what rate is the volume increasing when the radius is 3 feet?

6. If the edge of a cube be measured and found to be 8 inches and if an error of $\frac{1}{20}$ inch has been made, what is the approximate error in the calculated volume?

7. If an error of ·1 % is made in measuring the radius of a sphere, find approximately the percentage error in the calculated volume.

69. The result obtained in § 68 that if $y = x^3$

$$\Delta y = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

may be geometrically illustrated as follows.

The edge of a cube is x ins. and is increased by Δx ins. On inspection of Fig. 28 the volume added will be seen to be made

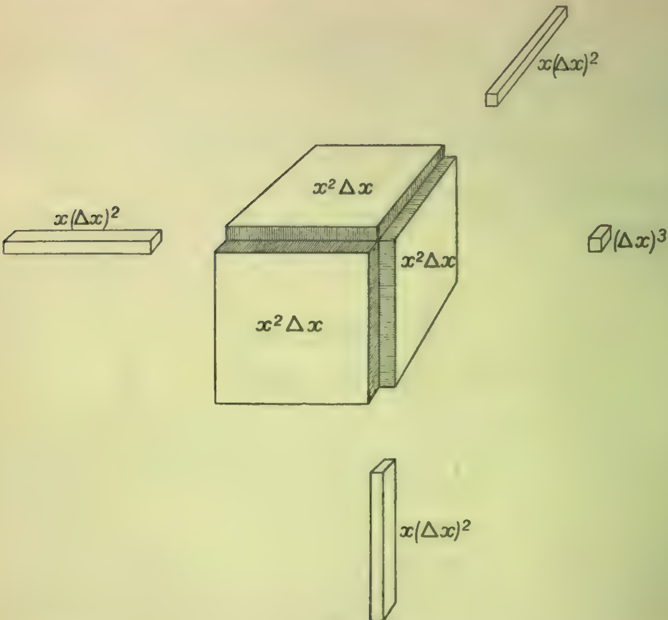


Fig. 28.

up of 3 slabs each of volume $x^2\Delta x$, 3 bars each of volume $x(\Delta x)^2$ and a cubical block of volume $(\Delta x)^3$.

The ratio of a bar to a slab is $\frac{\Delta x}{x}$, and the ratio of the small block to a bar is $\frac{\Delta x}{x}$.

When we say that the increase in volume is approximately $3x^2\Delta x$ we omit the bars and the small block. Suppose for example the edge of the original cube is 10 ins. and that it is increased by ·1 in. [i.e. $x=10$, $\Delta x = \cdot 1$, so that $\frac{\Delta x}{x} = \frac{1}{100}$.]

$$3x^2\Delta x = 300 \times \cdot 1 = 30,$$

$$3x(\Delta x)^2 = 30 \times \cdot 01 = \cdot 3,$$

$$(\Delta x)^3 = \cdot 001.$$

So that we take 30 cub. ins. as the increase in volume instead of 30·301.

70. Another illustration of the fact that if $y = x^3$, then $\frac{dy}{dx} = 3x^2$.

If $x = 3$, $y = 27$.

Take an increased value of x and find the increased value of y as shewn in the table:

$x+\Delta x$	$y+\Delta y$	Δx	Δy	$\frac{\Delta y}{\Delta x}$
3·5	42·875	·5	15·875	31·750
3·2	32·768	·2	5·768	28·84
3·1	29·791	·1	2·791	27·91
3·01	27·270901	·01	·270901	27·0901
3·001	27·027009001	·001	·27009001	27·009001

and $$3x^2 = 3 \times 3^2 = 27.$$

Thus as Δx is made smaller $\frac{\Delta y}{\Delta x}$ continually approaches $3x^2$ and can be made as near to it as we like if we take Δx small enough.

71. Sign of $\frac{dy}{dx}$. In § 50 $\frac{dp}{dV}$ turned out to be negative because p decreased when V increased.

Generally: Suppose y = any function of x.

If $\dfrac{\Delta y}{\Delta x}$ is positive Δx and Δy have the same sign; i.e. an increase in x produces an increase in y and a decrease in x produces a decrease in y.

Suppose the graph of $y=f(x)$ be drawn, P being the point (x, y) and Q $(x+\Delta x, y+\Delta y)$. Then if Δx and Δy are both positive P and Q will be placed as in (1), if both negative as in (2). (Fig. 29.)

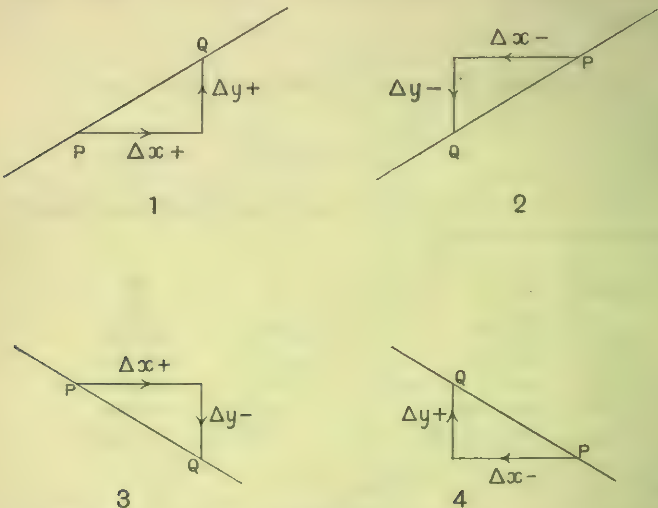


Fig. 29.

In each case the chord PQ has a positive gradient.

But if Δx be + and Δy −, P and Q will be placed as in (3), if Δx − and Δy +, as in (4).

In each case the chord PQ has a negative gradient.

So, if $\dfrac{dy}{dx}$ is positive for a given value of x, x and y are

both increasing or both decreasing, but if $\frac{dy}{dx}$ is negative, x is increasing and y decreasing or vice versa.

In the graph if $\frac{dy}{dx}$ is positive at the point (x, y) the gradient of the tangent is positive and the curve in the neighbourhood of P is shaped like (5) or (6), but if $\frac{dy}{dx}$ is negative, the shape is like (7) or (8). (Fig. 30.)

Fig. 30.

EXERCISES. XIII.

1. Find from first principles $\frac{dy}{dx}$ when $y=\frac{1}{x}$.

Illustrate your result by making a table like that in § 70. [Use a table of reciprocals.]

2. Find from first principles $\frac{dy}{dx}$ when $y=x+\frac{1}{x}$.

What is $\frac{dy}{dx}$ when $x=1$ and when $x=2$?

Draw the graph of $y=x+\frac{1}{x}$ between $x=\frac{1}{2}$ and $x=3$, and illustrate your answer by reference to this graph.

3. What is the equation of the tangent at the point $(2, 2\frac{1}{2})$ on the curve $y=x+\frac{1}{x}$?

4. Find from first principles $\frac{ds}{dt}$ when $s=3+2t+t^2$.

If a body move in a straight line, so that its distance from a fixed point at the end of t secs. is given by $s=3+2t+t^2$, find its speed at the end of (i) 3 seconds, (ii) 10 seconds.

5. Find from first principles $\frac{dy}{dx}$:

$$\text{(i) if } y=\frac{7}{x^2}, \quad \text{(ii) if } y=2x^3+3.$$

Find the equations of the tangent and normal to each of the curves $y=\frac{7}{x^2}$ and $y=2x^3+3$ at the point corresponding to $x=1$.

6. Find from first principles $\frac{dy}{dx}$ if $y=\sqrt{x}$.

$$\left[\text{Use } \sqrt{x+h}-\sqrt{x}=\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{\sqrt{x+h}+\sqrt{x}}=\frac{h}{\sqrt{x+h}+\sqrt{x}}.\right]$$

7. Find from first principles $\frac{dy}{dx}$ when $y=k$ (a constant).

8. From first principles, shew that if $y=x^{\frac{2}{3}}$ then $\frac{dy}{dx}=\frac{2}{3}x^{-\frac{1}{3}}$.

$$\left[\text{Put } y^3=x^2 \text{ and get } \frac{\Delta y}{\Delta x}=\frac{2x+\Delta x}{3y^2+3y\,.\,\Delta y+(\Delta y)^2}.\right]$$

9. From first principles, shew that if $x^2+y^2=a^2$, then $\frac{dy}{dx}=-\frac{x}{y}$.

$\left[\text{Get } \frac{\Delta y}{\Delta x}=-\frac{2x+\Delta x}{2y+\Delta y}.\right]$ Interpret this result geometrically.

CHAPTER III

DIFFERENTIATION OF x^n

72. In future instead of saying that "$\frac{dy}{dx}$ is the quantity which $\frac{\Delta y}{\Delta x}$ continually approaches as Δx continually approaches zero, and from which we can make it differ by as little as we please by making Δx small enough" we say that "$\frac{dy}{dx}$ is the **limit** of $\frac{\Delta y}{\Delta x}$ $\left(\text{or Lt. } \frac{\Delta y}{\Delta x}\right)$ when Δx is indefinitely diminished," but it must be constantly borne in mind that this is merely an abbreviated form of the longer statement.

73. A sign which has been recently introduced into mathematical text-books enables us to compress this phrase still further. The sign is $\rightarrow$. $z \rightarrow a$ means that z continually approaches a and can be brought as near to a as we please. Thus in § 68 we have $\frac{\Delta y}{\Delta x} = 3x^2 + 3x \,.\, \Delta x + (\Delta x)^2$ and we may say that as $\Delta x \rightarrow 0$, $\frac{\Delta y}{\Delta x} \rightarrow 3x^2$, meaning that as Δx approaches zero, $\frac{\Delta y}{\Delta x}$ continually approaches $3x^2$ and can be made as near to $3x^2$ as

we please if Δx be brought near enough to zero. The statements in inverted commas in § 72 may be conveniently written

$$\text{“}\frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Lt.}} \frac{\Delta y}{\Delta x}\text{”}$$

so that when we have found that as $\Delta x \to 0$, $\frac{\Delta y}{\Delta x} \to 3x^2$ we may say $\frac{dy}{dx} = 3x^2$.

74. If $y = x^n$, find $\frac{dy}{dx}$.

Let x receive a small increment Δx and let the corresponding increment in y be Δy.

Then $$y + \Delta y = (x + \Delta x)^n,$$

and $$y = x^n,$$

$$\therefore\ \Delta y = (x + \Delta x)^n - x^n.$$

[Now it is proved in algebra that if $h < x$

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots$$

for all values of n: also that starting from any term and taking any number of terms however large, the sum never exceeds a finite limit.]

$$\therefore\ \Delta y = nx^{n-1}(\Delta x) + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \text{terms containing higher powers of } (\Delta x)$$

$$= nx^{n-1}\Delta x + (\Delta x)^2 \times \mathsf{L}, \text{ where } \mathsf{L} \text{ is a finite quantity.}$$

$$\therefore\ \frac{\Delta y}{\Delta x} = nx^{n-1} + \mathsf{L}\,.\,\Delta x.$$

$$\therefore\ \text{as } \Delta x \to 0,\ \frac{\Delta y}{\Delta x} \to nx^{n-1},$$

or $$\frac{dy}{dx} = nx^{n-1}.$$

75. The proof of the Binomial Theorem for indices which are not positive integers is a very difficult piece of mathematics, which will probably never be met with by many readers of this book. For those who prefer a proof that if $y=x^n$, $\frac{dy}{dx}=nx^{n-1}$, which does not depend on this theorem, such a proof is subjoined.

76. We shall first establish the following theorem :—

For all values of n, positive and negative, integral and fractional, as z approaches nearer and nearer to 1, $\frac{z^n-1}{z-1}$ approaches nearer and nearer to n and can be made as near to n as we like if z be brought near enough to 1, or shortly:

$$\operatorname*{Lt.}_{z\to 1}\frac{z^n-1}{z-1}=n.$$

(1) Let n be a positive integer.

Then by actual division

$$\frac{z^n-1}{z-1}=z^{n-1}+z^{n-2}+z^{n-3}+\ldots+z+1.$$

There are n terms on the right, each of which can be made as near to 1 as we please if z is brought near enough to 1.

$$\therefore \text{ as } z\to 1,\ \frac{z^n-1}{z-1}\to n,$$

or

$$\operatorname*{Lt.}_{z\to 1}\frac{z^n-1}{z-1}=n.$$

e.g.

$$\frac{1{\cdot}1^3-1}{1{\cdot}1-1}=\frac{{\cdot}331}{{\cdot}1}=3{\cdot}31,$$

$$\frac{1{\cdot}01^3-1}{1{\cdot}01-1}=\frac{{\cdot}030301}{{\cdot}01}=3{\cdot}0301,$$

etc., and in this case $n=3$.

(2) Let n be a positive fraction, say $n=\frac{p}{q}$ where p and q are positive integers.

Also let $z^{\frac{1}{q}} = u$, $\therefore z = u^q$ and $z^{\frac{p}{q}} = u^p$.

$$\therefore \frac{z^n - 1}{z - 1} = \frac{u^p - 1}{u^q - 1} = \frac{u^p - 1}{u - 1} \bigg/ \frac{u^q - 1}{u - 1}.$$

As $z \to 1$, so does u.

Since p and q are positive integers,

$$\therefore \text{ as } u \to 1, \frac{u^p - 1}{u - 1} \to p \text{ and } \frac{u^q - 1}{u - 1} \to q.$$

$$\therefore \operatorname*{Lt.}_{z \to 1} \frac{z^n - 1}{z - 1} = \frac{p}{q} \text{ or } n.$$

e.g.
$$\frac{1 \cdot 1^{\frac{3}{4}} - 1}{1 \cdot 1 - 1} = \frac{1 \cdot 0741 - 1}{1 \cdot 1 - 1} = \frac{\cdot 0741}{\cdot 1} = \cdot 741,$$

$$\frac{1 \cdot 01^{\frac{3}{4}} - 1}{1 \cdot 01 - 1} = \frac{1 \cdot 00749 - 1}{1 \cdot 01 - 1} = \frac{\cdot 00749}{\cdot 01} = \cdot 749,$$

and in this case $n = \frac{3}{4} = \cdot 75$.

(3) Let n be negative $= -m$ where m is positive.

$$\therefore \frac{z^n - 1}{z - 1} = \frac{z^{-m} - 1}{z - 1} = \frac{\frac{1}{z^m} - 1}{z - 1} = -\frac{1}{z^m} \cdot \frac{z^m - 1}{z - 1}.$$

Since m is positive, $\operatorname*{Lt.}_{z \to 1} \frac{z^m - 1}{z - 1} = m$, and $\operatorname*{Lt.}_{z \to 1} z^m = 1$.

$$\therefore \operatorname*{Lt.}_{z \to 1} \frac{z^n - 1}{z - 1} = -m \text{ or } n.$$

e.g.
$$\frac{1 \cdot 1^{-\frac{1}{2}} - 1}{1 \cdot 1 - 1} = \frac{\cdot 9535 - 1}{\cdot 1} = -\cdot 465,$$

$$\frac{1 \cdot 01^{-\frac{1}{2}} - 1}{1 \cdot 01 - 1} = \frac{\cdot 99504 - 1}{\cdot 01} = -\cdot 496,$$

and in this case $n = -\frac{1}{2} = -\cdot 5$.

77. Now with the notation of § 74,

$$\frac{\Delta y}{\Delta x}=\frac{(x+\Delta x)^n-x^n}{(x+\Delta x)-x}$$

$$=\frac{x^n\left\{\left(\frac{x+\Delta x}{x}\right)^n-1\right\}}{x\left\{\frac{x+\Delta x}{x}-1\right\}}$$

$$=x^{n-1}\cdot\frac{z^n-1}{z-1}$$

where z stands for $\frac{x+\Delta x}{x}$.

Now as $\Delta x \to 0$, $\frac{x+\Delta x}{x}$ or $z \to 1$ and the limit of $\frac{z^n-1}{z-1}$ is therefore n.

$$\therefore \underset{\Delta x\to 0}{\text{Lt.}}\ \frac{\Delta y}{\Delta x}=nx^{n-1},$$

or

$$\mathbf{\frac{dy}{dx}=nx^{n-1}.}$$

This is a most important formula and includes all the results we have obtained hitherto.

e.g. If $y=x^3$, $\frac{dy}{dx}=3x^2$.

If $y=\frac{1}{x}=x^{-1}$, $\frac{dy}{dx}=(-1)\,x^{-2}=-\frac{1}{x^2}$.

If $y=\sqrt{x}=x^{\frac{1}{2}}$, $\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$.

EXERCISES. XIV.

1. Write down $\frac{dy}{dx}$ when

$$y=x^5,\ x^{20},\ x^{\frac{1}{3}},\ x^{-3},\ \sqrt[4]{x},\ \frac{1}{x^4},\ x^{1\cdot 8},\ x^{\cdot 4},\ x,\ 5,$$

$$x^{\frac{3}{2}},\ x^{\frac{3}{4}},\ x^{-\frac{1}{2}},\ x^{-6},\ \sqrt[3]{x^5},\ \frac{1}{\sqrt{x}},\ \frac{1}{\sqrt[3]{x^2}}.$$

2. In the above piece of work (§ 74 or 77) what difference would it make if $y=kx^n$ where k is a constant, i.e. some number independent of x?

3. What difference would it make if $y=x^n+c$ where c is a constant?

78. The results of the last two examples are very important.

(i) Taking $$\mathbf{y = kx^n}$$

we should have $$y + \Delta y = k\,(x + \Delta x)^n.$$

$$\therefore\ \Delta y = k\left[(x + \Delta x)^n - x^n\right].$$

i.e. the value of Δy in this case is k times its value in the case when $y = x^n$.

The rest of the work is the same to the end, except that this factor k remains, and we get eventually

$$\mathbf{\frac{dy}{dx} = knx^{n-1}.}$$

(ii) Taking $$\mathbf{y = x^n + c,}$$

we should have $$y + \Delta y = (x + \Delta x)^n + c.$$

$$\therefore\ \Delta y = (x + \Delta x)^n - x^n$$

exactly as when $$y = x^n.$$

$\therefore$ in this case $\mathbf{\dfrac{dy}{dx} = nx^{n-1}.}$

EXERCISES. XV.

1. Write down $\dfrac{dy}{dx}$ when $y = 3x^5$, $2x^{15}$, $\dfrac{3}{x}$, $5\sqrt{x}$, $2x^2+3$, $\dfrac{3}{x^2}-8$, $4x^{1\cdot41}$, $\dfrac{3}{x^{\cdot091}}$, $\dfrac{1}{3}x^3$, $\dfrac{\sqrt{x}}{5}$, x^3+6, $7\sqrt{x}+5$.

2. Write down $\dfrac{ds}{dt}$ if $s = \dfrac{1}{3}t^2 + 5$.

3. Write down $\dfrac{dp}{dv}$ if $pv^{1\cdot4} = 500$.

4. Write down $\dfrac{dR}{d\theta}$ if $R = k\,(1 + a\theta)$ where k and a are constants.

79. Graphical illustration of the result of § 78 (i). Draw on the same sheet, using the same scales

$$y = x^2 \ldots (1) \text{ and } y = 3x^2 \ldots (2). \quad \text{[Fig. 31.]}$$

In (1) $\dfrac{dy}{dx} = 2x$. In (2) $\dfrac{dy}{dx} = 6x$.

So that at the point (2, 4) on (1), the gradient of the curve is 4 and at the point (2, 12) on (2), the gradient of the curve is 12.

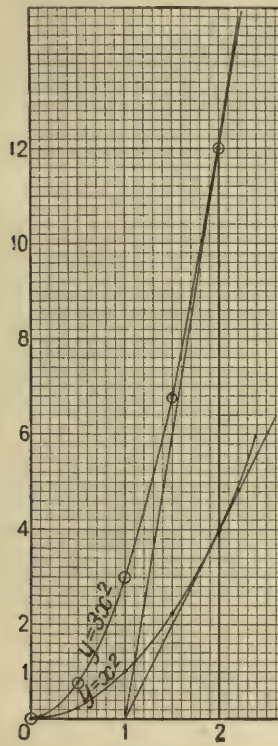


Fig. 31.

i.e. the gradient of (2) at any point is 3 times the gradient of (1) at the point with the same abscissa, i.e. at the point vertically under it.

EXERCISES. XVI.

1. Shew that the tangents at (2, 4) and (2, 12) to the curves $y=x^2$ and $y=3x^2$ meet OX in the same point.

2. Shew that if a tangent be drawn to $y=x^2$ at the point (c, c^2) and a tangent to $y=kx^2$ at the point (c, kc^2) the gradient of the second tangent is k times that of the first and that the two tangents meet OX in the same point.

3. Make and prove similar statements about $y=x^3$ and $y=kx^3$ and generally about $y=x^n$ and $y=kx^n$.

80. Kinematical illustration. Suppose that two bodies are moving in the same straight line and that the distance of the first from a fixed point in the line at the end of a given time is determined by the equation $s=t^2$, and that the distance of the second body from the same point is given by $s=3t^2$; then the speed of the first at any instant is 3 times that of the second at the same instant; in fact, in any interval of time however small, the second body moves 3 times as far as the first.

81. Graphical illustration of the result of § 78 (ii).

Draw on the same sheet, using the same scales [Fig. 32]

$$y=x^2\ldots(1) \text{ and } y=x^2+2\ldots(2),$$

then in each case
$$\frac{dy}{dx}=2x,$$

so that at the point (2, 4) on (1) the gradient is 4 and at the point (2, 6) on (2) the gradient is 4.

i.e. the gradient at two corresponding points is the same.

This is obvious from the fact that (2) may be obtained by sliding (1) bodily parallel to OY through a distance 2.

82. Kinematical illustration. Suppose that two bodies are moving in the same straight line and that their distances from a fixed point in the line at the end of a given time are

given by the equations $s = t^2$ and $s = t^2 + 2$ respectively, then their speeds are the same at every instant, for the second body is always the same distance, 2 feet, ahead of the first.

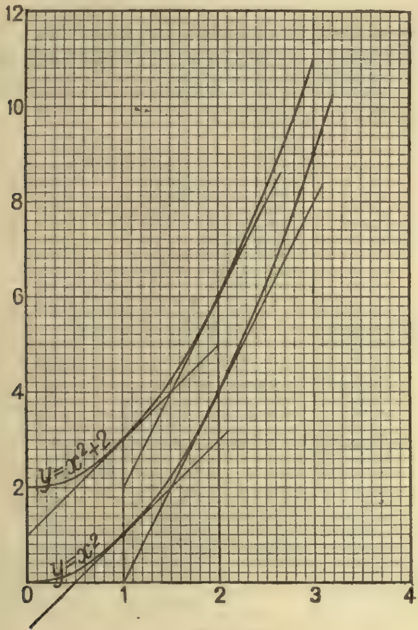


Fig. 32.

83. We found in § 39 that if $s = 5 + 3t + 2t^3$,

$$\frac{ds}{dt} = 3 + 6t^2.$$

Now if $s = 5$, $\frac{ds}{dt} = 0$; if $s = 3t$, $\frac{ds}{dt} = 3$; if $s = 2t^3$, $\frac{ds}{dt} = 6t^2$;

so that in this case $$\frac{ds}{dt} = \frac{d(5)}{dt} + \frac{d(3t)}{dt} + \frac{d(2t^3)}{dt}.$$

Generally if $y=u+v$ where u and v stand for two functions of x whose differential coefficients we know,

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx},$$

for if x receive a small increment Δx, there will be a resulting small increment in u which we may call Δu, and a small increment in v which we may call Δv, and Δy, the increment in y, is equal to the sum of these.

In fact $$y+\Delta y=u+\Delta u+v+\Delta v,$$

so that $$\Delta y=\Delta u+\Delta v.$$

$$\therefore \frac{\Delta y}{\Delta x}=\frac{\Delta u}{\Delta x}+\frac{\Delta v}{\Delta x}.$$

If Δx be made smaller and smaller $\frac{\Delta u}{\Delta x}$ and $\frac{\Delta v}{\Delta x}$ can be made to differ by as little as we please from $\frac{du}{dx}$ and $\frac{dv}{dx}$ respectively.

$\therefore \frac{\Delta y}{\Delta x}$ can be made to differ from $\frac{du}{dx}+\frac{dv}{dx}$ by as little as we please; i.e.

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}.$$

Similarly if $y=u+v+w+\ldots$, $\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+\ldots$.

EXERCISES. XVII.

1. Write down $\frac{dy}{dx}$ (i) if $y=3+2x+4x^2$, (ii) if $y=5-\frac{6}{x}+\frac{1}{x^2}$.

2. Write down $\frac{ds}{dt}$ if $s=5t^3-\frac{1}{2}t^2+1$.

3. Draw on the same sheet the graphs of $y=3$, $y=2x$, $y=4x^2$, and $y=3+2x+4x^2$, and verify that at any point [say where $x=1$] the gradient of the fourth is equal to the sum of the gradients of the other three.

4. If $y=ax^2+bx+c$, shew that $\left(\frac{dy}{dx}\right)^2=4ay+b^2-4ac$.

84. *Example.* Find (i) the gradients of the tangents to the curve

$$y = 2x^2 - 3x - 1$$

at the points where $\quad x = -1, 0, 1, 2$;

(ii) at what point the tangent is parallel to OX.

Also find y when $x = -1, 0, 1, 2$ and make use of all this information to draw the graph of $y = 2x^2 - 3x - 1$ between $x = -1$ and $x = 2$.

$$y = 2x^2 - 3x - 1.$$

$$\therefore \ \frac{dy}{dx} = 4x - 3.$$

$\therefore$ we get

x	-1	0	1	2
y	4	-1	-2	1
$\frac{dy}{dx}$	-7	-3	1	5

Also the tangent is parallel to OX when the gradient is 0, i.e. when $x = \frac{3}{4}$, and for this value of x, $y = -2\frac{1}{8}$.

Plot the points $(-1, 4)$ $(0, -1)$ $(1, -2)$ $(2, 1)$ $\left(\frac{3}{4}, -2\frac{1}{8}\right)$ and through these points draw lines whose gradients are -7, -3, 1, 5, 0 respectively.

Then our curve must go through all these points and touch all these lines. [Fig. 33.]

EXERCISES. XVIII.

1. If $y = x^3 - 3x$, what is $\frac{dy}{dx}$? Find the values of y and $\frac{dy}{dx}$ corresponding to $x = -2, -1, 0, 1, 2$, and draw the curve $y = x^3 - 3x$ between $x = -2$ and $x = 2$.

2. If $y = x^4 - 4x^3 + 4x^2 - 3$ what is $\frac{dy}{dx}$? Find the values of y and $\frac{dy}{dx}$ when $x = -1, 0, 1, 2, 3$, and draw the curve $y = x^4 - 4x^3 + 4x^2 - 3$ between $x = -1$ and $x = 3$.

3. If $y=2x^3-9x^2+12x-3$ find $\dfrac{dy}{dx}$.

Also find the values of y and $\dfrac{dy}{dx}$ when $x=0, 1, 2, 3$.

From these data, draw the graph of $y=2x^3-9x^2+12x-3$ between $x=0$ and $x=3$.

4. Find the points on $y=x^3-3x^2+2x$ at which the gradient is zero and draw the graph between $x=0$ and $x=3$.

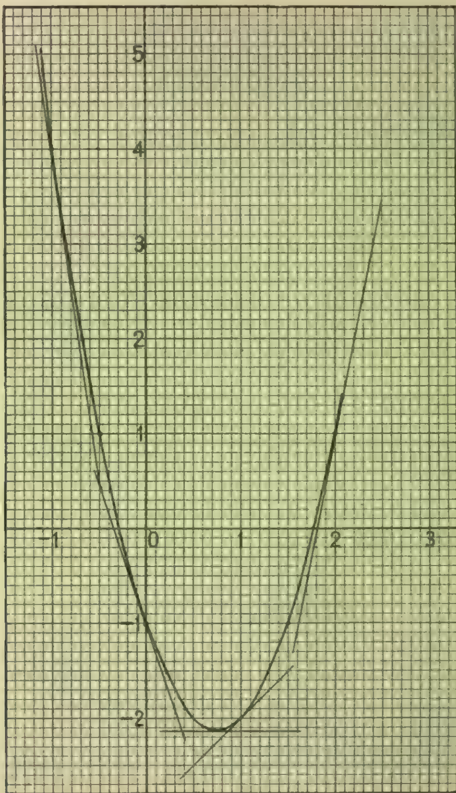


Fig. 33.

85. In the following examples we make use of the following theorem :—

If a line whose gradient is m be drawn through the point (h, k) its equation is

$$\mathbf{y - k = m\,(x - h).}$$

That this is so is easily seen, for

(1) The equation represents a straight line since it is of the first degree in x and y.

(2) It is satisfied when $x = h$ and $y = k$ and therefore passes through (h, k).

(3) Its gradient is m since it may be written

$$y = mx + (k - mh).$$

e.g. the line through (3, 5) whose gradient is 4 is

$$y - 5 = 4\,(x - 3) \quad \text{or} \quad \mathbf{y = 4x - 7}.$$

86. *Ex.* 1. Find the equations of the tangent and normal (perpendicular to the tangent) at the point for which $x = 3$ on the curve $y = 2x + 3x^3$.

We have $\dfrac{dy}{dx} = 2 + 9x^2$.

When $x = 3$, $y = 87$ and $\dfrac{dy}{dx} = 83$.

$\therefore$ the tangent is the line through (3, 87) whose gradient is 83.

$\therefore$ its equation is $\quad y - 87 = 83\,(x - 3)$

or
$$\mathbf{y = 83x - 162.}$$

The normal is perpendicular to the tangent.

$\therefore$ its gradient is $\qquad -\dfrac{1}{83}$,

and its equation $\qquad y - 87 = -\dfrac{1}{83}(x - 3)$

or
$$\mathbf{x + 83y = 7224.}$$

Ex. 2. In the curve $y=\frac{x^2}{4a}$, P is the point where $x=h$. Find the equations of the tangent and normal at P.

We have $y=\frac{x^2}{4a}$, $\quad\therefore \frac{dy}{dx}=\frac{x}{2a}$.

$\therefore$ when $x=h$, $y=\frac{h^2}{4a}$ and $\frac{dy}{dx}=\frac{h}{2a}$.

$\therefore$ the tangent is the line through $\left(h, \frac{h^2}{4a}\right)$ whose gradient is $\frac{h}{2a}$.

$\therefore$ its equation is $y-\frac{h^2}{4a}=\frac{h}{2a}(x-h)$,

or
$$\mathbf{y=\frac{h}{2a}x-\frac{h^2}{4a}} \quad \text{.......................(1)}.$$

The normal is the line through $\left(h, \frac{h^2}{4a}\right)$ whose gradient is $-\frac{2a}{h}$.

$\therefore$ its equation is $y-\frac{h^2}{4a}=-\frac{2a}{h}(x-h)$,

or
$$\mathbf{y=-\frac{2a}{h}x+\left(\frac{h^2}{4a}+2a\right)} \quad \text{...............(2)}.$$

Ex. 3. From the results of Ex. 2 we can deduce some interesting properties of the curve $y=\frac{x^2}{4a}$. (Fig. 34.)

First we notice that the curve passes through the origin, and is symmetrical about OY, i.e. the same value of y is obtained by putting $x=h$ as by putting $x=-h$. Also y is always positive.

Let P be the point $\left(h, \frac{h^2}{4a}\right)$, and let the tangent and normal at P meet OY in T and G and let PN be perpendicular to OY.

In (1) put $x=0$, $\quad\therefore\ y=-\dfrac{h^2}{4a}$,

i.e. co-ordinates of T are $\left(0, -\dfrac{h^2}{4a}\right)$ or $OT=\dfrac{h^2}{4a}$, but $ON=\dfrac{h^2}{4a}$.

$$\therefore\ OT = ON.$$

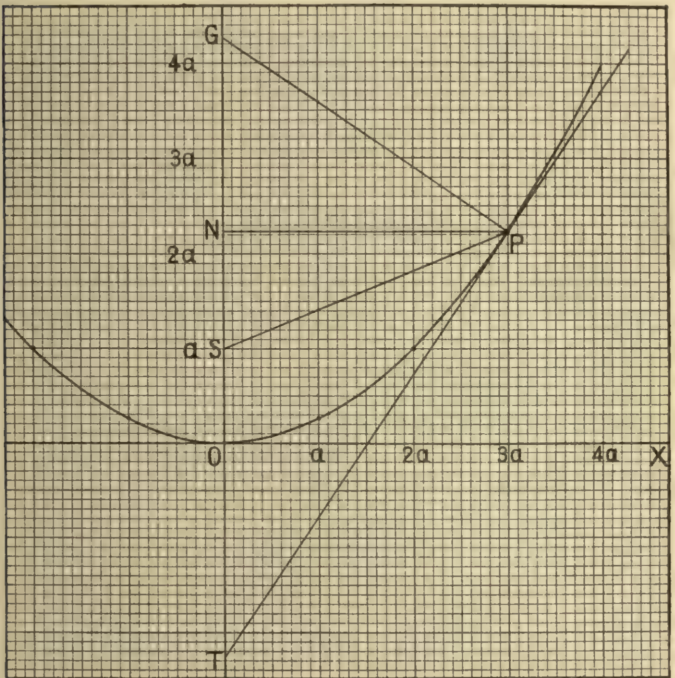


Fig. 34.

In (2) put $x=0$. $\quad\therefore\ y=\dfrac{h^2}{4a}+2a$, i.e. the co-ordinates of G are $\left(0, \dfrac{h^2}{4a}+2a\right)$ or $OG=\dfrac{h^2}{4a}+2a$, but $ON=\dfrac{h^2}{4a}$.

$\therefore$ NG $= 2a$, i.e. NG is the same for all positions of P.

Again if S is the point $(0, a)$,

$$\text{ST} = \text{SO} + \text{OT} = a + \frac{h^2}{4a},$$

and
$$\text{SG} = \text{OG} - \text{OS} = a + \frac{h^2}{4a}.$$

$$\therefore\ \text{ST} = \text{SG},$$

i.e. S is the centre of the semi-circle on TG as diameter.

But this semi-circle passes through P, since TPG is a right angle.

$$\therefore\ \text{ST} = \text{SG} = \text{SP}.$$

EXERCISES XIX.

1. Taking the unit as $\frac{1''}{2}$ along each axis and $a=1$, what is the equation of the curve in Exs. 2 and 3, § 86? Draw the curve from $x=-4$ to $x=+4$.

Let P be the point at which $x=3a$.

Take OT = ON and verify that TP is the tangent at P.

Take NG $=2a$ and verify that GP is perpendicular to TP.

2. If P be any point on $y=\frac{x^2}{4a}$, S the point $(0, a)$, KK′ the line $y=-a$, PM perpendicular to KK′, prove SP = PM.

[This curve is called a *parabola*, S is its *focus* and KK′ its *directrix*.]

3. In your figure to Question 1 shew the focus and directrix, and verify SP = PM.

4. If the tangent at any point P on $y=\frac{x^2}{4a}$ meets the directrix in Z, shew that PSZ is a right angle.

5. Find the equations of the tangent and normal at the point (1, 1) to the curve $y=x^3$. If the tangent meet OX and OY in T and t respectively and the normal meet OX and OY in G and g respectively, find the coordinates of T, t, G, g.

6. Find the equations of the tangent and normal at the point where $x=2$ on the curve $y=3x^4$: also at the point where $x=3$ on the curve $xy=1$.

7. P is the point $\left(h, \frac{h^3}{a^2}\right)$ on the curve $y=\frac{x^3}{a^2}$. PN and Pn are perpendicular to OX, OY. The tangent at P meets OX, OY in T, t and the normal meets OX, OY in G, g. Prove

$$\text{OT}=\frac{2}{3}\text{ON},\ \text{O}t=2\text{O}n,\ \text{ON}\,.\,\text{NG}=3\,.\,\text{O}n^2,\ \text{O}n\,.\,ng=\frac{1}{3}\text{ON}^2.$$

8. Find the equations of the tangent and normal at each of the points on the curve $y^2=x^3$ where $x=4$. Draw a figure.

9. Find the equations of the tangent and normal to the curve

$$y=7x^2-2x+6,$$

at the point where $x=3$.

10. The equation of a curve is $x+y=a^{\frac{1}{4}}x^{\frac{3}{4}}$.

Find the gradient of the tangent at the point where $x=16a$. Find where this tangent meets the line $x+y=0$.

11. Find the equations of the tangent and normal at the point where $x=2$ on the curve $y=3x+x^2$.

If the point be P and if the tangent and normal meet the y-axis in T and G respectively find the area of $\triangle$TPG.

The functional notation.

87. The fact that y is a function of x is sometimes expressed in the form $y=f(x)$, e.g. if $y=x^2-7x+8$, $f(x)$ is x^2-7x+8. $f(2)$ means the result of substituting 2 for x in this expression,

i.e. $$f(2)=2^2-7\times 2+8=-2.$$

Similarly $$f(0)=8,\quad f(-3)=38,$$

$$f(2p)=4p^2-14p+8,\ f(z^2+2)=(z^2+2)^2-7(z^2+2)+8$$
$$=z^4-3z^2-2, \text{ etc.}$$

Other notations sometimes used are F(x), $\phi(x)$, etc.

It is very important to bear in mind that $f(x)$ does not stand for the product of x and some quantity f, but is really an instruction written in shorthand to perform a certain operation or series of operations on x.

Thus in the above example where $f(x)$ stands for x^2-7x+8, the symbol f placed before a number means—square the number, subtract 7 times the number and add 8.

EXERCISES. XX.

1. If $f(x)=\dfrac{x+2}{2x-3}$ find $f(1)$, $f(0)$, $f(x+5)$.

2. If $\phi(x)=3x^3-2x^2+\dfrac{4}{x}$, find $\phi(6)$, $\phi(y+1)$.

3. If $f(x)=x+\sin x$ [x being in radians] find $f\left(\dfrac{\pi}{2}\right)$, $f(1)$, $f(0)$.

88. If $y=f(x)$, the differential coefficient of y with respect to x may be written $\dfrac{dy}{dx}$ or $\dfrac{df(x)}{dx}$ or $f'(x)$.

Thus if
$$f(x)=3x^3-9x^2+7x-10,$$
$$f'(x)=9x^2-18x+7.$$

$f'(x)$ is itself a function of x, and we may therefore evaluate such expressions as

$$f'(3),\ f'(0),\ f'(z+3), \text{ etc.}$$

e.g.
$$f'(3)=9\times 3^2-18\times 3+7$$
$$=34.$$

With our usual notation we should say that this was the value of $\dfrac{dy}{dx}$ when $x=3$.

Thus we may say that the gradient at any point of the curve $y=f(x)$ is $f'(x)$, or that the gradient at the point for which $x=a$ is $f'(a)$.

Similarly if $s=f(t)$,

$$\frac{ds}{dt} \text{ may be written } f'(t),$$

and we may say that $f'(t)$ gives the speed at the end of time t.

Higher differential coefficients.

89. If $y=f(x)$ then $\frac{dy}{dx}$ or $f'(x)$ is also a function of x and may therefore itself be differentiated.

The result of this differentiation may be written

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} \text{ or } \frac{df'(x)}{dx}.$$

These are abbreviated into $\frac{d^2y}{dx^2}$ and $f''(x)$ respectively.

$$\text{e.g. if} \quad y=f(x)=2x^4-3x^3+2x,$$

$$\frac{dy}{dx}=f'(x)=8x^3-9x^2+2,$$

$$\frac{d^2y}{dx^2}=f''(x)=24x^2-18x.$$

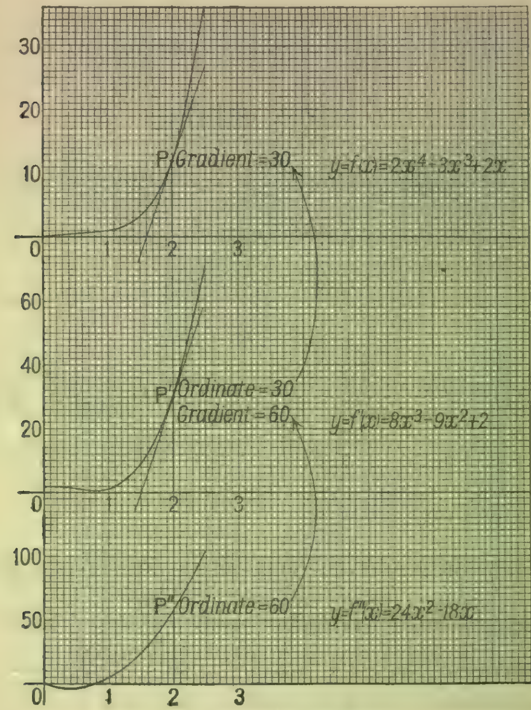
90. Just as $\frac{dy}{dx}$ or $f'(x)$ tells us the rate of increase of y or $f(x)$ with respect to x, so $\frac{d^2y}{dx^2}$ or $f''(x)$ tells us the rate of increase of $\frac{dy}{dx}$ or $f'(x)$ with respect to x.

Considered in relation to the graph $y=f(x)$, $\frac{dy}{dx}$ or $f'(x)$ gives the gradient at any point on the graph; $\frac{d^2y}{dx^2}$ or $f''(x)$ gives the rate at which the gradient is increasing with respect to x.

If in the last illustration we put $x=2$, we get

$$f(2)=2\times 2^4-3\times 2^3+2\times 2$$
$$=12.$$
$$f'(2)=8\times 2^3-9\times 2^2+2$$
$$=30.$$
$$f''(2)=24\times 2^2-18\times 2$$
$$=60.$$

i.e. at the point (2, 12) on the curve $y = 2x^4 - 3x^3 + 2x$, y is increasing 30 times as fast as x, or the gradient of the curve is 30, and the gradient is increasing 60 times as fast as x. (Fig. 35.)



Figs. 35, 36, 37.

Or we may say that at the point (2, 30) on the curve

$$y = 8x^3 - 9x^2 + 2 \text{ [i.e. } f'(x)]$$

which is called the first derived curve of $y = 2x^4 - 3x^3 + 2x$, the gradient is 60. (Fig. 36.)

91. If we draw the graphs

$$y=f(x),$$
$$y=f'(x),$$
$$y=f''(x),$$

the number of units in the ordinate of $y=f'(x)$ corresponding to any value of x gives the gradient at the corresponding point on $y=f(x)$, and the number of units in the ordinate of $y=f''(x)$ gives the gradient at the corresponding point on $y=f'(x)$.

e.g., looking at Figs. 35, 36, 37 which give the graphs of $y=f(x)$, $y=f'(x)$, $y=f''(x)$ for the particular case when

$$f(x) \equiv 2x^4 - 3x^3 + 2x,$$

P, P′, P″ are the points corresponding to $x=2$. The ordinate of P″ is 60 units and 60 is the gradient at P′. The ordinate of P′ is 30 units and 30 is the gradient at P.

92. If $s=f(t)$, $\frac{ds}{dt}$ or $f'(t)$ (sometimes written $\dot{s}$) gives the speed at end of time t.

If we call this v we have seen [v. Ex. (5) p. 51] that $\frac{dv}{dt}$ gives the acceleration at end of time t.

Now $\frac{dv}{dt}$ is $\frac{d^2s}{dt^2}$ or $f''(t)$.

Thus $\frac{d^2s}{dt^2}$ or $f''(t)$ [sometimes written $\ddot{s}$] gives the acceleration at end of time t.

The graphs of $f(t)$, $f'(t)$, $f''(t)$ are respectively the space-time, speed-time and acceleration-time graphs.

93. We get interesting special cases when $f(x)$ is of the 1st or 2nd degree.

e.g. if
$$\left.\begin{aligned} f(x) &= 2x^2 - 3x + 1, \\ f'(x) &= 4x - 3, \\ \text{and} \quad f''(x) &= 4. \end{aligned}\right\}$$

The graphs of $y=f(x)$, $y=f'(x)$, $y=f''(x)$ are shewn in Fig. 38.

$y=f'(x)$ is a straight line, and the gradient at every point is the same.

$y=f''(x)$ is a straight line, the ordinate at every point being the same, i.e. it is a straight line parallel to OX.

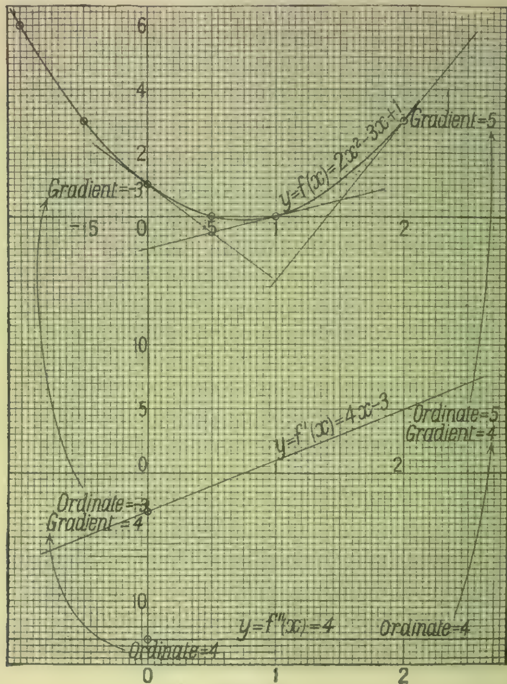


Fig. 38.

In the general case,

if $\qquad f(x)=ax^2+bx+c,$ where a, b, c are constant,

$$f'(x)=2ax+b,$$

and $\qquad f''(x)=2a.$

If $$s = f(t) = at^2 + bt + c,$$
$$v = f'(t) = 2at + b,$$
and $$a = f''(t) = 2a.$$

i.e. if s be a quadratic function of t, the acceleration is constant.

In this case the speed-time graph is a straight line and the acceleration-time graph a straight line parallel to OX.

If $$f(x) = 4x + 3,$$
$$f'(x) = 4,$$
$$f''(x) = 0.$$

The graphs are shewn in Fig. 39.

$y = f(x)$ is a straight line,

$y = f'(x)$ is a straight line parallel to OX,

$y = f''(x)$ is OX.

Generally if $f(x) = ax + b$, where a and b are constant,
$$f'(x) = a,$$
$$f''(x) = 0.$$

If $$s = f(t) = at + b,$$
$$v = f'(t) = a,$$
$$a = f''(t) = 0.$$

i.e. if s be a linear function of t the speed is constant and the acceleration zero.

EXERCISES. XXI.

1. If $f(x) = 2x + \frac{1}{x}$, find $f'(x)$ and $f''(x)$.

Also find $f(2)$, $f'(2)$, $f''(2)$.

Draw the graphs $y = f(x)$, $y = f'(x)$, $y = f''(x)$ between $x = -1$ and $x = +1$.

Make a statement like that in § 91, taking $x = \frac{1}{2}$.

2. If $s = 3 - 4t + 2t^2 + t^3$ find the speed and acceleration at the end of 3 seconds.

Draw the space-time, speed-time and acceleration-time graphs.

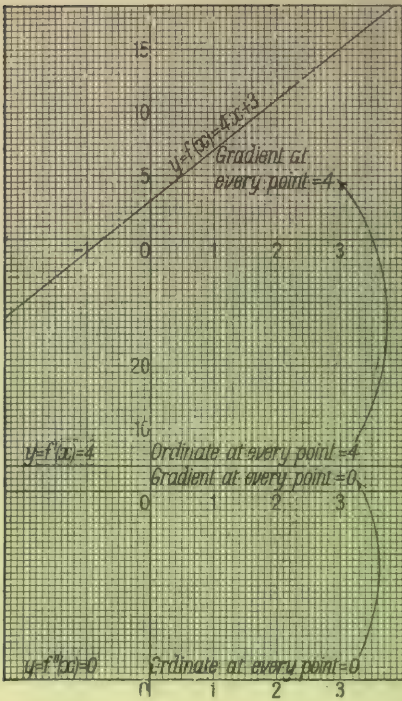


Fig. 39.

94. We have seen (§ 71) that if $\frac{dy}{dx}$ is positive, y is increasing as x increases, but that if $\frac{dy}{dx}$ is negative y is decreasing as x increases.

Similarly if $\frac{d^2y}{dx^2}$ or $\frac{d\left(\frac{dy}{dx}\right)}{dx}$ is positive, $\frac{dy}{dx}$ is increasing as x increases, but if $\frac{d^2y}{dx^2}$ is negative, $\frac{dy}{dx}$ is decreasing as x increases.

Now if between P and Q a curve has a shape as in Fig. 40 (*a*) it is clear that as x increases from OM to ON, y increases and also the gradient or $\frac{dy}{dx}$ increases. [The positive directions of the x and y axes are supposed to be as usual 'right' and 'up' respectively.]

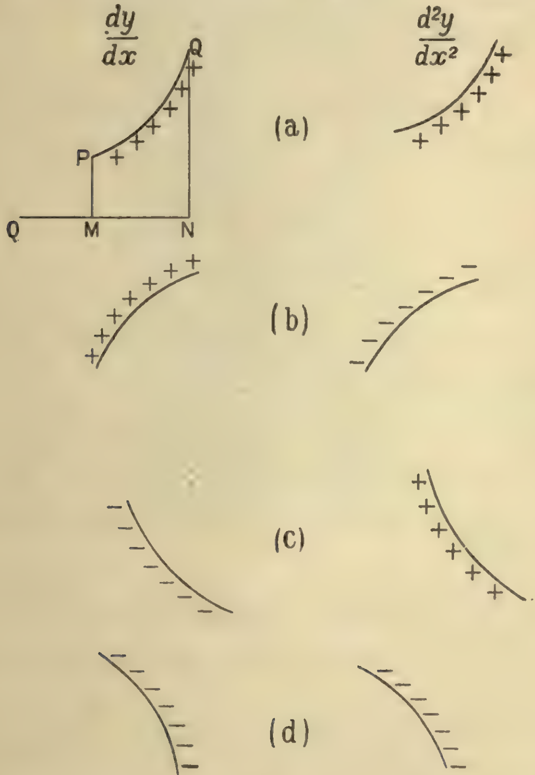


Fig. 40.

i.e. for all points between P and Q

$$\frac{dy}{dx} \text{ is + and } \frac{d^2y}{dx^2} \text{ is + .}$$

Shew similarly that corresponding to a shape like Fig. 40 (*b*)

$$\frac{dy}{dx} \text{ is } + \text{ and } \frac{d^2y}{dx^2} \text{ is } -.$$

For Fig. 40 (*c*) $\quad \dfrac{dy}{dx}$ is $-$ and $\dfrac{d^2y}{dx^2}$ is $+$.

For Fig. 40 (*d*) $\quad \dfrac{dy}{dx}$ is $-$ and $\dfrac{d^2y}{dx^2}$ is $-$.

95. e.g. consider the curve $y = 2x^3 - 9x^2 + 12x - 3$.

We have
$$\frac{dy}{dx} = 6x^2 - 18x + 12$$
$$= 6(x-1)(x-2),$$

and
$$\frac{d^2y}{dx^2} = 12x - 18.$$

For all values of x between 2 and 3, $\dfrac{dy}{dx}$ is $+$ and $\dfrac{d^2y}{dx^2}$ is $+$.

$\therefore$ between $x = 2$ and $x = 3$ the curve is of the form shewn in Fig. 40 (*a*).

For all values of x between 0 and 1, $\dfrac{dy}{dx}$ is $+$ and $\dfrac{d^2y}{dx^2}$ is $-$.

$\therefore$ between $x = 0$ and $x = 1$ the curve is of the form shewn in Fig. 40 (*b*).

For all values of x between $1\frac{1}{2}$ and 2, $\dfrac{dy}{dx}$ is $-$ and $\dfrac{d^2y}{dx^2}$ is $+$.

$\therefore$ between $x = 1\frac{1}{2}$ and $x = 2$ the curve is of the form shewn in Fig. 40 (*c*).

For all values of x between 1 and $1\frac{1}{2}$, $\dfrac{dy}{dx}$ is $-$ and $\dfrac{d^2y}{dx^2}$ is $-$.

$\therefore$ between $x = 1$ and $x = 1\frac{1}{2}$ the curve is of the form shewn in Fig. 40 (*d*).

[See Fig. 41.]

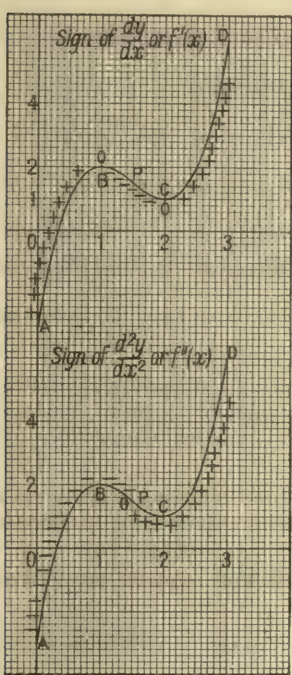


Fig. 41.

EXERCISES. XXII.

What is the form of the following curves?

1. $y=2x^2-5x+7$: (i) between $x=2$ and $x=3$,
(ii) ,, $x=0$ and $x=1$.

2. $y=6+12x-x^3$: (i) ,, $x=1$ and $x=2$,
(ii) ,, $x=-1$ and $x=-2$,
(iii) ,, $x=3$ and $x=4$.

3. $y=x-\frac{1}{x}$: (i) when x is positive,
(ii) ,, ,, negative.

MISCELLANEOUS EXAMPLES ON CHAPTER III.

1. Find from first principles

(i) $\frac{dy}{dx}$ if $y=x^{\frac{2}{3}}$. (ii) $\frac{dp}{dv}$ if $pv^2=k$ (a constant).

2. Find the equations of the tangent and normal to the curve $y=\frac{1}{x}$ at each of the points $\left(2, \frac{1}{2}\right)$, $\left(5, \frac{1}{5}\right)$ and in each case find the area of the triangle formed by the tangent and the axes of co-ordinates.

3. Do the same for the point $\left(c, \frac{1}{c}\right)$. State a geometrical property of the tangent at any point of the curve.

4. P is the point (3, 9) on the curve $y=x^2$. Find the equations of the tangent and normal at P, and the co-ordinates of the points T, G where they meet the y-axis. If PN is perpendicular to the y-axis, find the lengths of NT and NG.

5. A body moves in a straight line in such a way that its distance (s ft.) from a fixed point in the line at the end of t seconds is given by

$$s=7-5t+9t^2.$$

Find its distance from the fixed point, its speed and acceleration at the end of 5, 6, 7 seconds.

6. The same if $\quad s=3+t-5t^2+7t^3.$

7. In the curve $y=2x+3x^2$, find the gradient at the point P where $x=2$.

Find also the gradient of the chord joining P to Q, Q being the point where $x=2{\cdot}01$.

The tangent at P and the line through P parallel to OX meet the ordinate of Q in T and R respectively. Find the percentage error made in using RT instead of RQ. Explain the connection between the statements RT = RQ approximately and $\Delta y=\frac{dy}{dx}\cdot\Delta x$ approximately.

8. Find the equation of the tangent to the curve

$$y=3x^4-4x^3-12x^2+5,$$

at the point where $x=3$.

Also find the co-ordinates of the points at which the tangent is parallel to the x-axis.

9. P is the point on $y=cx^n$ (c a constant) at which $x=h$. The tangent and normal at P meet the x-axis in T, G, and the y-axis in t, g. PN, Pn are perpendicular to OX, OY. Find

$$\text{(i) } \frac{\mathrm{OT}}{\mathrm{ON}}, \quad \text{(ii) } \frac{\mathrm{O}t}{\mathrm{O}n}, \quad \text{(iii) } \frac{\mathrm{ON}\,.\,\mathrm{NG}}{\mathrm{O}n^2}, \quad \text{(iv) } \frac{\mathrm{O}n\,.\,ng}{\mathrm{ON}^2}.$$

10. A trough is in the shape of a right prism with its ends equilateral triangles. The length of the trough is 10 feet. It contains water, which leaks at the rate of 1 cubic foot per minute. Find in ins./sec. the rate at which the level is sinking when the depth of water is (i) 3 inches, (ii) 1 foot.

11. A current C of electricity is changing according to the law

$$\mathrm{C}=20+21t-14t^2,$$

where t is seconds.

The voltage V is such that

$$\mathrm{V}=\mathrm{RC}+\mathrm{L}\,.\,\frac{d\mathrm{C}}{dt},$$

where $\mathrm{R}=0{\cdot}5$, and $\mathrm{L}=0{\cdot}01$.

Find V when $t=2$.

12. A man runs a given total distance in such a way that the time, t secs., he takes to run any part, s feet, of the distance is given by the equation $t=as^n$, a and n being constants. Compare his average speed over the whole distance with his speed at the end.

13. A body of constant mass m is moving with variable speed v. If K is its kinetic energy and M its momentum shew that $\frac{d\mathrm{K}}{dv}=\mathrm{M}$. State what your units are.

14. If $p\mathrm{V}=k$ (a constant) shew that $\frac{dp}{d\mathrm{V}}=-\frac{p}{\mathrm{V}}$,

and if $p\mathrm{V}^n=k$, $$\frac{dp}{d\mathrm{V}}=-n\frac{p}{\mathrm{V}}.$$

15. The coefficient of cubical expansion of a substance at temperature $\theta°$ is the rate of increase of volume per unit increase of temperature.

The volume (V c.c.) of a gram of water at $\theta°$ C. is given by

$$\mathrm{V}=1+a\,(\theta-4)^2$$

where $$a=8{\cdot}38\times10^{-6}.$$

Find $\frac{d\mathrm{V}}{d\theta}$ and hence get the coefficient of cubical expansion of water at 0° C. and at 20° C.

16. The specific heat of a substance at temperature $\theta°$ is the rate of increase of Q per unit increase in θ, where Q is the number of heat units required to raise the temperature of 1 gm. from some standard temperature to $\theta°$.

The total heat required to raise the temperature of 1 gm. of water from 0° to $\theta°$ C. is given by

$$Q = \theta + 2 \times 10^{-5}\,\theta^2 + 3 \times 10^{-7}\,\theta^3.$$

Find the specific heat of water at 80° C.

17. For diamond the formula is

$$Q = \cdot 0947\theta + \cdot 000497\theta^2 - \cdot 00000012\theta^3.$$

Find the specific heat of diamond at 80° C.

18. If $y = 2x^4 - 2x^3 - x^2 + 1$ find y when $x = -1,\ -\frac{1}{4},\ 0,\ 1,\ 2.$

Also find $\frac{dy}{dx}$ when $x = -1$, 2 and find what values of x make $\frac{dy}{dx} = 0$.

Using all this information draw the graph of

$$y = 2x^4 - 2x^3 - x^2 + 1$$

between $x = -1$ and $x = 2$.

19. In a certain case of straight-line motion, the number of feet in the distance from a fixed point, at a given instant t seconds after the start, is given by the formula

$$s = 3t + 4t^3.$$

Calculate

(i) The average speed between the end of the 2nd and the end of the 4th second.

(ii) The mean of the speeds at the instants 2 and 4 seconds after the start.

(iii) The speed 3 seconds after the start.

20. Find the equations of the tangent and normal to $y = x^3$ at the point (2, 8).

If P is the point (2, 8) and if the tangent and normal meet the x-axis in T and G and PN is perpendicular to OX, find the lengths NT, NG.

21. P is the point $\left(h, \frac{h^2}{4a}\right)$ on the parabola $y = \frac{x^2}{4a}$; PT is the tangent at P. PM is perpendicular to the line $y = -a$. S is the point $(0, a)$. Prove SM is perpendicular to PT and meets it on the x-axis.

22. A vessel containing water is in the form of an inverted hollow cone with vertical angle 90°. If the depth of water be x feet, what is the volume of water?

If water flows in at the rate of 1 cubic foot per minute, at what rate is the level of water rising when the depth is 2 feet?

23. Find the co-ordinates of the point of intersection of the tangents to $y=3x^2+5x-7$ at the points where $x=2$ and $x=5$.

Verify that this point and the mid-point of the chord joining the two points of contact lie on a line parallel to OY.

24. The shape of the vertical section of a hill is given (approximately) by the graph of $y=\cdot 05x^2+\cdot 25x$ from $x=0$ to $x=6$ and by the graph of

$$y=-4\cdot 2+1\cdot 65x-\frac{1}{15}x^2$$

from $x=6$ to $x=12$.

Calculate the ordinates when $x=1, 2, 3, \ldots 12$ and the gradient at each of the corresponding points. Shew that when $x=6$ the gradient is the same for each part of the hill. Draw the figure taking $\frac{1''}{2}$ as the unit each way.

25. Find the equation of that tangent to the parabola $y=3x^2$ which is parallel to $3x-2y=7$.

26. If $s=3t-7t^2+16t^3$, s being the number of feet and t the number of seconds, find the formula for the acceleration at the end of t seconds. If 8 lbs. is the mass of the moving body, what is the force acting on it at the end of 4 seconds?

27. A body is moving in a straight line and its distance (s feet) from a fixed point at the end of t seconds is given by the following table:

t	0	1	2	3	4	5	6
s	0	7	22	45	76	115	162

Find graphically the speed at the end of 3 seconds.

28. A wheel in t seconds rotates through $(5t+4t^3)$ radians from some standard position. Find its angular velocity and acceleration after 5 seconds.

29. The side walls of a truck are vertical and its section by a plane parallel to the side walls is a trapezium, upper side 14 feet, lower side 10 feet, depth 3 feet. The width of the truck is 4 feet. It is originally full of sand (sp. gr. 1·5) which leaks out in such a way that the depth of sand diminishes at a constant rate of 1 inch per minute. Find the mass of sand in the truck after t seconds. If the truck is made to move in a straight line with a uniform speed of 15 miles an hour, what is the momentum of the sand in the truck at the end of t seconds? What is the resultant force on the truck at the end of t seconds? [Remember force is given by rate of change of momentum.]

Find the force at the end of (i) 5 minutes, (ii) 20 minutes.

30. If
$$y=\frac{1+x}{2+x},$$
find graphically the value of $\frac{dy}{dx}$ when $x=1$.

CHAPTER IV

MAXIMA AND MINIMA

96. Plot roughly the curve $y = 2x^3 - 9x^2 + 12x - 3$ between $x = 0$ and $x = 3$.

You will get a curve as in Fig. 42.

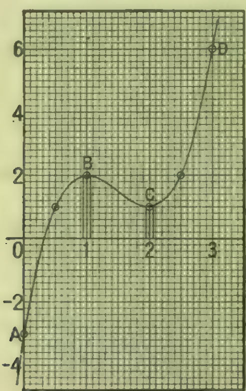


Fig. 42.

Between A and B, $\dfrac{dy}{dx}$ is positive,

,, B and C ,, negative,

,, C and D ,, positive again,

or in other words as we advance from left to right, y increases up to B and then begins to decrease; as we come to C, y goes on decreasing and then begins to increase. (§ 71.)

Points like B and C are called **turning points** and at B y is said to have a **maximum** value, and at C a **minimum** value.

Notice, that although we call the value of y at B a maximum, yet this is not by any means the greatest value y can have. A maximum point is one where the ordinate (representing the function) stops increasing and begins to decrease; in other words the value of y at a maximum point is greater than the value of y at neighbouring points on either side, and similarly at a minimum point the value of y is less than the value of y at neighbouring points on either side.

97. Now suppose we wish to find the exact position of these turning points B and C.

At a turning point the tangent to the curve is parallel to OX, i.e. the gradient of the curve is zero, or

$$\frac{dy}{dx}=0.$$

$\therefore$ **to find the turning points, we get $\frac{\mathbf{dy}}{\mathbf{dx}}$ and find the values of x which make $\frac{\mathbf{dy}}{\mathbf{dx}}=0$.**

e.g. in the case above

$$\frac{dy}{dx}=6(x^2-3x+2)=6(x-1)(x-2),$$

$$\therefore \quad \frac{dy}{dx}=0 \text{ when } x=1 \text{ or } 2.$$

When $x=1$, $y=2$; when $x=2$, $y=1$.

$\therefore$ (1, 2) and (2, 1) are the turning points.

98. In order to find whether a turning point corresponds to a maximum or minimum we may either draw a rough graph or proceed as follows:

when $x=1$, $y=2$;

when $x=1+h$, $y=2(1+h)^3-9(1+h)^2+12(1+h)-3$

$$=2-3h^2+2h^3.$$

Now if h is a small fraction, $2h^3$ is small compared with $3h^2$, and the sign of $-3h^2+2h^3$ is that of the first term, and is therefore negative whether h is positive or negative.

Thus when $x=1$, $y=2$; but if x is slightly greater or less than 1, $y<2$. $\therefore$ $x=1$ corresponds to a maximum value of y.

Similarly, shew that when $x=2$, $y=1$; but if x is slightly greater or less than 2, $y>1$.

$\therefore$ $x=2$ corresponds to a minimum value of y.

99. There is an exceptional case which we shall consider later. Suppose

$$y=3x^4-16x^3+30x^2-24x+5.$$

$$\frac{dy}{dx}=12x^3-48x^2+60x-24$$

$$=12(x-1)(x-1)(x-2),$$

and $$\frac{dy}{dx}=0 \text{ when } x=1 \text{ or } 2.$$

If $x=1$, $y=-2$.

If $x=1+h$, $y=-2-4h^3+3h^4$.

If h is a small positive fraction, $y<-2$.

If h is a small negative fraction, $y>-2$.

Thus when $x=1$, $y=-2$.

If x is slightly greater than 1, $y<-2$.

If x is slightly less than 1, $y>-2$.

So that here there is neither maximum nor minimum.

The graph is shewn in the first part of Fig. 56, p. 130, and the point P is called a point of inflexion.

100. We may state the results of § 97:

"**The values of x which make f (x) a maximum or minimum are roots of f′ (x) = 0.**"

In the case considered

$$f(x) = 2x^3 - 9x^2 + 12x - 3,$$
$$f'(x) = 6x^2 - 18x + 12.$$

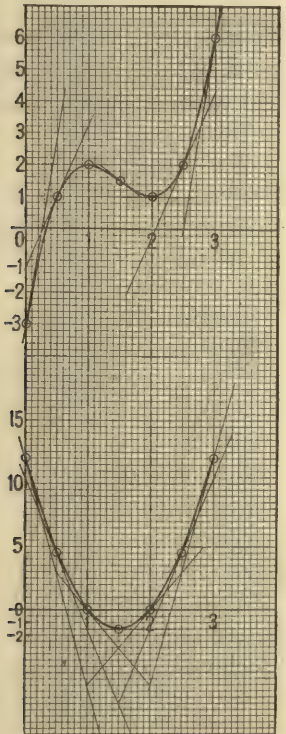


Fig. 43.

If we plot the curves $y = f(x)$ and $y = f'(x)$ taking the same x-scale each time, the points where $y = f'(x)$ cuts the x-axis will correspond to turning points on $y = f(x)$. (Fig. 43.)

Another Test for Maxima and Minima.

101. It is evident from Fig. 44 that **if we advance from left to right, i.e. in the positive x-direction, f′ (x) or $\frac{dy}{dx}$ changes from + to − as we pass through a maximum point and from − to + as we pass through a minimum point.** (See § 71.)

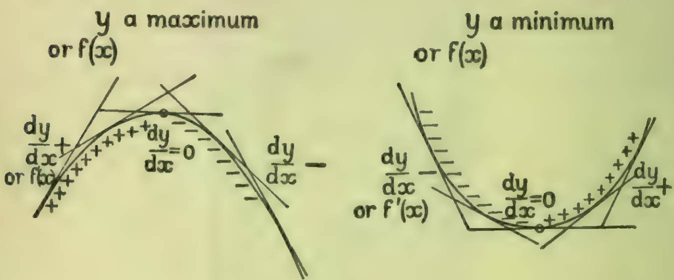


Fig. 44.

Consider the example in § 96.

$$f(x) = 2x^3 - 9x^2 + 12x - 3,$$
$$f'(x) = 6x^2 - 18x + 12$$
$$= 6(x-1)(x-2).$$

The values of x, which make $f'(x) = 0$, are 1 and 2.

If x is slightly less than 1, $f'(x)$ is + for $\left.\begin{aligned} x-1 \text{ is } -\\ \text{and } x-2 \text{ is } -\end{aligned}\right\}.$

If x is slightly more than 1, $f'(x)$ is − for $\left.\begin{aligned} x-1 \text{ is } +\\ x-2 \text{ is } -\end{aligned}\right\}.$

This may be abbreviated conveniently:

$$\left.\begin{array}{lll}\text{If} & x=1-, & f'(x)=(-)(-)=+\\ \text{If} & x=1+, & f'(x)=(+)(-)=-\end{array}\right\}.$$

$\therefore$ as we pass from left to right through the point (1, 2) $f'(x)$ changes from + to $-$.

$\therefore$ (1, 2) is a maximum point.

Similarly shew that (2, 1) is a minimum point.

102. If we apply this test to the case in § 99,

where $$f(x)=3x^4-16x^3+30x^2-24x+5,$$

we have $$f'(x)=12(x-1)(x-1)(x-2).$$

The values of x which make $f'(x)=0$ are 1 and 2.

$$\left.\begin{array}{lll}\text{If} & x=1-, & f'(x)=(-)(-)(-)=-\\ \text{If} & x=1+, & f'(x)=(+)(+)(-)=-\end{array}\right\},$$

i.e. $f'(x)$ does not change sign as we pass through the point $(1, -2)$, and this point is neither a maximum nor a minimum.

EXERCISES. XXIII.

1. Find the turning points of the curve $y=x^3-3x$ and plot enough of the curve to shew these points.

2. Shew that $y=x^3+3x$ has no turning points. Plot it between $x=-2$ and $x=+2$. Also plot $y=f'(x)$.

3. Find the turning points of the curve

$$y=4x^3+5x^2-12x+5.$$

4. Find the turning points of the curve

$$y=x^4-2x^2+1$$

and plot between $x=-2$ and $x=+2$.

Also plot $y=f'(x)$ between the same limits.

5. Shew that $y=x^4+2x^2+1$ has only one turning point and plot between $x=-2$ and $x=+2$. Plot $y=f'(x)$.

6. Find the turning points on

$$y=3x^4-4x^3-12x^2+5.$$

Plot between $x=-2$ and $x=+2$. Plot $y=f'(x)$.

7. Shew that $y=x+\dfrac{1}{x}$ has two turning points and that the maximum point is lower than the minimum.

8. Find the turning point of $y=2x^2-3x+5$. Is it a maximum or a minimum point?

9. Shew that $y=ax^2+bx+c$ has always one turning point and that it is a maximum or minimum according as a is negative or positive.

10. Shew that $y=ax^3+bx^2+cx+d$ has two turning points or none.

103. *Examples.* (1) The stiffness of a rectangular beam varies as bd^3 where b is the breadth and d the depth. Find the breadth and depth of the stiffest beam the cross section of which has a perimeter of 4 feet.

Let the depth be x feet.

$\therefore$ the breadth is $(2-x)$ feet, and the beam will be stiffest when $(2-x)x^3$ is greatest. Call this y.

$$y=(2-x)x^3$$
$$=2x^3-x^4,$$
$$\therefore\ \frac{dy}{dx}=6x^2-4x^3.$$

For a maximum or minimum value of y

$$\frac{dy}{dx}=0.$$
$$\therefore\ x=0 \text{ or } \frac{3}{2}.$$

$x=0$ obviously corresponds to a zero value of y.

$${}^*x=\frac{3}{2} \text{ gives a maximum.}$$

* It is clear that there must be a maximum between $x=0$ and $x=2$, for $y=0$ when $x=0$ or 2 and is positive for intermediate values of x, or we may say

$$\frac{dy}{dx}=4x^2\left(\frac{3}{2}-x\right).$$

When $\quad x=\frac{3}{2}-,\ \frac{dy}{dx}=(+)(+)=+;$

when $\quad x=\frac{3}{2}+,\ \frac{dy}{dx}=(+)(-)=-;$

$$\therefore\ x=\frac{3}{2} \text{ gives a maximum value of } y.$$

Thus the stiffest beam is $1\frac{1}{2}$ ft. deep and $\frac{1}{2}$ ft. broad.

Fig. 45 shews the graph of $y=(2-x)x^3$ between $x=0$ and $x=2$.

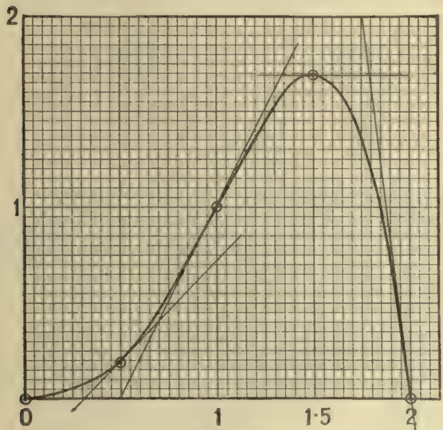


Fig. 45.

(2) The volume of an open cylindrical canister is to be 100 cubic inches. Find the most economical dimensions, i.e. the dimensions which give the least surface area.

Let the radius of base be x ins. and the height h ins.

Then surface area $=(\pi x^2+2\pi xh)$ sq. ins.

We want the least value of this.

At present the surface involves two variables x and h but we are given

$$\pi x^2 h = 100,$$

$$\therefore\ \pi h = \frac{100}{x^2}.$$

$$\therefore \text{ Surface area } = \pi x^2 + 2x \cdot \frac{100}{x^2}$$

$$= \pi x^2 + \frac{200}{x}.$$

Calling this y, we have

$$\frac{dy}{dx} = 2\pi x - \frac{200}{x^2}.$$

For a maximum or minimum value of y, $\frac{dy}{dx} = 0$.

$$\therefore\ x^3 = \frac{100}{\pi}, \qquad \therefore\ x = \sqrt[3]{\frac{100}{\pi}} = 3{\cdot}17,$$

and
$$h = \frac{100}{\pi} \times \frac{\pi^{\frac{2}{3}}}{100^{\frac{2}{3}}} = \sqrt[3]{\frac{100}{\pi}} = 3{\cdot}17,$$

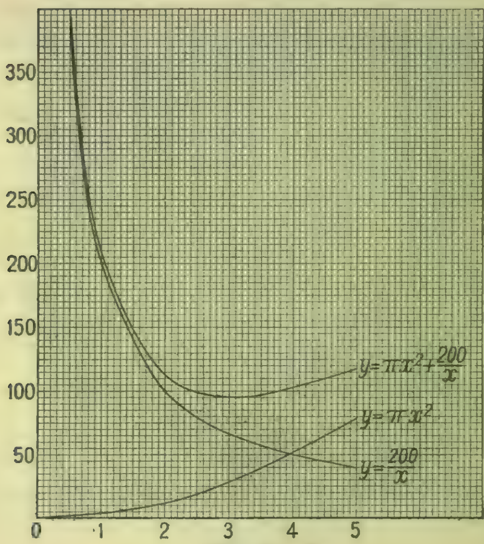


Fig. 46.

The corresponding value of y is $30\sqrt[3]{10\pi} = 94{\cdot}6$. [Shew that this is a minimum value.]

$\therefore$ The canister has the radius of the base and the height each 3·17 ft. and 94·6 sq. ins. of material are required to make it.

[That $x = h$ can be seen as follows:

$$\left.\begin{aligned}\pi x^2 h &= 100\\ \pi x^3 &= 100\end{aligned}\right\}\ \therefore\ x = h.]$$

Fig. 46 shews the graph of

$$y = \pi x^2 + \frac{200}{x}.$$

(3) Find two factors of 24 whose sum is a minimum.

Let the factors be x and $\frac{24}{x}$ and call their sum y.

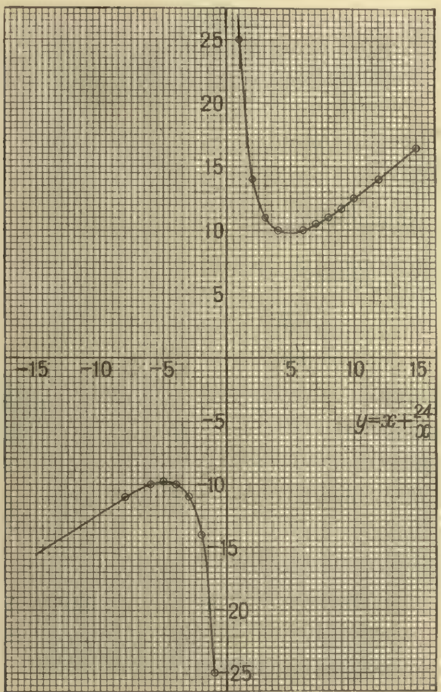


Fig. 47.

Thus
$$y = x + \frac{24}{x},$$
$$\therefore \frac{dy}{dx} = 1 - \frac{24}{x^2},$$
$$\therefore \frac{dy}{dx} = 0 \text{ when } x^2 = 24.$$

$\therefore$ y is a maximum or minimum when
$$x = \pm\sqrt{24}$$
$$= \pm 4{\cdot}899 \ldots$$

$$\left.\begin{array}{l}\text{If } x = 4{\cdot}899, \text{ sum of factors} = 9{\cdot}798\ldots \\ \text{If } x = 5, \text{ other factor is } 4{\cdot}8, \text{ sum} = 9{\cdot}8\end{array}\right\} \begin{array}{l}\therefore \text{ Sum is a minimum} \\ \text{when } x = \sqrt{24}.\end{array}$$

Similarly shew that sum is a maximum when
$$x = -\sqrt{24}.$$

Thus the required factors are 4·899, 4·899.

Fig. 47 shews the graph of
$$y = x + \frac{24}{x}.$$

104. The following example illustrates a useful device.

A ship B is 75 miles due east of a ship A. B sails west at 9 knots and A south at 12 knots.

Find when their distance apart is least. Find also what the east distance is.

P, Q are positions after x hrs. [Fig. 48.]
$$\text{AP} = 75 - 9x = 3\,(25 - 3x) \text{ miles},$$
$$\text{AQ} = 12x \qquad = 3\,(4x),$$
$$\therefore\ \text{PQ}^2 = 9\,[(25 - 3x)^2 + (4x)^2]$$
$$= 9\,[625 - 150x + 25x^2]$$
$$= 225\,[25 - 6x + x^2].$$

Put $$y = x^2 - 6x + 25,$$

$$\therefore \frac{dy}{dx} = 2x - 6,$$

$\therefore$ y is a minimum when $x = 3$. [Why minimum?]

$\therefore$ least distance is after 3 hrs.

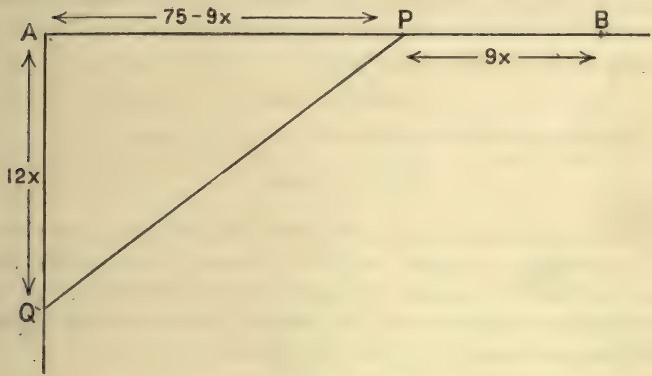


Fig. 48.

When $$x = 3, \; PQ^2 = 225 \times 16,$$

$$\therefore \; PQ = 60.$$

Instead of finding when PQ is a minimum we found when PQ^2 was a minimum. If we had expressed PQ in terms of x, we should have $15\sqrt{25 - 6x + x^2}$ and we do not yet know how to differentiate this.

105. Notice that we have to express the quantity, of which we wish to find the maximum or minimum value, as a function of one unknown.

In some cases it is first of all necessary to express it in terms of two or more unknowns, and then by means of given conditions to express all these unknowns in terms of one of them.

Thus in the second example the surface was written first as a function of x and h, then by means of the given relation, $\pi x^2 h = 100$, we were able to express h in terms of x, and so get the surface as a function of x only.

106. A common error in questions of maximum and minimum values arises from a confusion between variables and constants. Take, for instance, the second example of § 103.

If y sq. ins. is the surface area

$$y = \pi x^2 + 2\pi x h.$$

The error consists in saying that

$$\frac{dy}{dx} = 2\pi x + 2\pi h.$$

This would only be right if h were constant, i.e. if h remained the same when x changed, which is obviously not the case, for since the volume remains the same, h must increase when x decreases and decrease when x increases. In fact h is a function of x, being connected with x by the equation

$$\pi x^2 h = 100,$$

or

$$h = \frac{100}{\pi x^2},$$

and this value must be substituted for h before we proceed to get $\frac{dy}{dx}$.

EXERCISES. XXIV.

1. Find the number such that when it is added to its reciprocal the sum is a minimum.

2. The sum of the square and the reciprocal of a number is to be a minimum.

Find the number. Draw a graph shewing how this sum changes with the number (positive numbers only).

3. Divide a number a into two parts so that three times the square of one part plus four times the square of the other shall be a minimum.

4. Find the dimensions of the rectangle of greatest area with a perimeter of 10 feet.

Draw a graph shewing how the area changes with the length of the rectangle.

5. Prove that the greatest rectangle that can be inscribed in a given circle is a square.

6. A rectangle PQRS is inscribed in a triangle so that P lies in AB, Q in AC and R, S in BC. If $BC=a$, $QR=x$ and the altitude of the triangle $=h$, shew that $PQ=a\left(1-\frac{x}{h}\right)$. Shew that the rectangle of maximum area has its height one-half that of the triangle.

7. Find the dimensions of the greatest cylinder that can be inscribed in a cone of radius r and height h, the planes of the bases of the cone and cylinder being the same.

8. The surface of a hollow cylinder without top is 100 sq. ins. Find the maximum volume.

9. The volume of a solid cylinder is V cubic ins. Find its dimensions if the total surface is a minimum.

10. A sheet of tin is 6 feet by 4 feet. A square is cut out of each corner, and a tank made by bending up the projecting portions. Find the side of the squares cut out so that the volume of the tank may be a maximum.

11. The number of tons of coal consumed per hour by a certain ship is $0{\cdot}3+0{\cdot}001V^3$ where V knots is the speed. For a voyage of 1000 miles at V knots, find the total consumption of coal. Find for what speed the coal consumption is least.

Do the same for 100 miles.

12. A running track is in the form of a rectangle ABCD with semi-circles on AB, CD. If the perimeter is a quarter of a mile find the maximum area.

13. A window is in the form of a rectangle surmounted by a semi-circle. If the perimeter is 30 feet, find the dimensions so that the greatest possible amount of light may be admitted.

14. The strength (i.e. resistance to breaking) of a rectangular beam varies as bd^2 where b is breadth and d depth. Find the breadth and depth of the strongest beam that can be cut from a circular log of diameter 2 feet.

15. A shot is projected in vacuo with velocity u ft./sec. in a direction making an angle α with the horizontal. Its height above the ground at the end of t seconds is $tu \sin \alpha - \frac{1}{2} gt^2$. Find the greatest height and the time of reaching it.

16. A shot is projected with velocity u ft./sec. in a direction making an angle α with the horizontal. An inclined plane making an angle β with the horizontal passes through the point of projection. The distance of the shot from this plane at the end of t secs. is

$$tu \sin (\alpha - \beta) - \frac{1}{2} g \cos \beta \, . \, t^2.$$

Find its greatest distance from the plane and the time of reaching it.

17. By the G.P.O. regulations the combined length and girth of a parcel must not exceed 6 feet. Find the volume of the greatest parcel that can be sent

(i) in the shape of a rectangular solid with square ends,

(ii) in the shape of a cylinder,

(iii) in the shape of a cone.

18. An open tank whose base is a square has to contain 1000 cubic feet of water. Find the least cost of lining it with lead at 4*d.* per sq. ft.

19. A man has 108 square feet of iron which he is going to make into a rectangular tank, lidless, and such that the depth is to be double the breadth. What will be the length, breadth and depth of the tank constructed under these conditions which will hold the greatest volume of water?

20. Find the circular cylinder of largest volume which can be cut from a sphere of radius 6 inches, the plane ends being perpendicular to the axis.

21. Find the volume of the greatest cylinder that can be inscribed in a sphere of radius a.

What is the ratio of the height of the cylinder to the diameter of the base?

22. Find the angle of the circular sector of greatest area that can be made having a given perimeter a.

23. A dynamo is in two parts whose weights are x and y. The cost of the machine is £ $(y + 4x)$. The usefulness of the machine is proportional to $(x^2 + 3xy)$. Find the values of x and y for maximum usefulness in a dynamo worth £10.

24. The cost per hour of driving a steamer through water varies as the cube of the speed relative to the water. Shew that the most economical speed against a current of V miles/hour is $\frac{3}{2}$V. [If £C is cost per mile, find when $\frac{1}{C}$ is a maximum.]

25. Given 200 sq. feet of canvas, find the greatest conical tent that can be made out of it. [Get V^2 in terms of r and l (slant height).]

26. Given a circular sheet of paper. Find the angle of the sector which must be cut out so that the remainder may be folded to give a conical vessel of maximum volume.

27. Find the least area of canvas that can be used to construct a conical tent whose cubical capacity is 800 cubic feet.

28. Find the volume of the greatest right cone that can be described by the revolution about a side of a right-angled triangle of hypotenuse 2 feet.

29. There are n voltaic cells each of E.M.F. E volts and internal resistance r ohms; x cells are arranged in series and $\frac{n}{x}$ rows in parallel. The current C amperes that the battery will send through an external resistance R is given by

$$C=\frac{xE}{\frac{x^2r}{n}+R}.$$

Given $$n=20,\quad R=\cdot25,\quad r=\cdot2,$$

find how many cells should be arranged in series to give the maximum current.

30. A battery, internal resistance r ohms and E.M.F. E, sends a current through an external resistance R.

The power W given to the external circuit is given by

$$W=\frac{RE^2}{(R+r)^2}.$$

Given E and r find what the external resistance must be so that the power may be the greatest possible.

31. The power W given to an external circuit by a generator of internal resistance r ohms and E.M.F. E when the current is C amperes is given by

$$W=CE-C^2r.$$

If $$E=60,\quad r=\cdot3,$$

find C so that W may be a maximum.

32. The annual cost of giving a certain amount of electric light to a certain town, the voltage being V and the candle-power of each lamp C, is found to be

$$A = a + \frac{b}{V} \text{ for electric energy,}$$

and

$$B = \frac{m}{C} + \frac{nV^{\frac{2}{3}}}{C^{\frac{5}{3}}} \text{ for lamp renewals.}$$

The following figures are known when C is 10.

V	100	200
A	1500	1200
B	300	500

Find a, b, m, n. If $C = 20$ what value of V will give the minimum total cost?

33. Shew that the Mechanical Power of a motor is greatest when the efficiency is 50 %.

[Mec. Power $= e \,.\, Ca$; Efficiency $= \frac{e}{E}$; $E - e = Ca \,.\, Ra$. E and Ra are constants.]

34. A given weight W hangs from a point B of a straight horizontal lever ABC which can turn about A. The lever weighs w lbs. per foot length. If the length of AB is given, say a feet, find the length of lever for which the requisite vertical supporting force at C will be a minimum.

35. A piece of wire of length 6 feet is to be cut into six portions, two equal of one length and four equal of another. The two former are each bent into the form of a circle and these are held in parallel planes and fastened together by the four remaining pieces which are perpendicular to the planes of the circles. The whole thus forms a model of a cylinder. Calculate the lengths into which the wire must be divided so as to produce a cylinder of maximum volume.

36. In measuring electric current by a tangent galvanometer the percentage error due to a given small error in the reading is proportional to $\tan x + \cot x$. Find the value of x for which this is a minimum. [Put $\tan x = t$.]

37. A log is in the form of a frustum of a cone, 20 feet long, the diameters of the ends being 2 feet and 1 foot. Find the length of the greatest beam of square section that can be cut from it.

Do the same if the length is l feet and the diameters of the ends a and b feet.

What is the relation between a and b if the length of the greatest beam is the same as the length of the log?

38. Find the length of the shortest line which will divide a given triangle into two parts of equal area.

39. An electric current flows round a coil of radius a.

A small magnet is placed with its axis on the line perpendicular to the plane of the coil through its centre.

If x is the distance of the magnet from the plane of the coil, the force exerted on it by the current is proportional to

$$\frac{x}{(x^2+a^2)^{\frac{5}{2}}}.$$

Find x so that the force may be a minimum. [Put $x^2+a^2=y$.]

40. A tank standing on the ground is kept full of water to a depth a ft. Water issues from a small aperture at a depth h ft. below the surface with velocity $\sqrt{2gh}$ ft./sec.

Find h in order that the water may strike the ground at the greatest possible distance from the tank.

Another way of distinguishing between Maximum and Minimum.

107. If $f(x)$ is a function of x which is increasing as x increases, or decreasing as x decreases, **$f'(x)$ is positive** [§ 71], but if $f(x)$ is decreasing as x increases, or increasing as x decreases, **$f'(x)$ is negative.** [Fig. 49.]

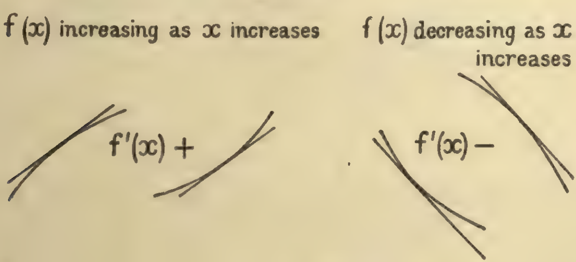


Fig. 49.

Now as we pass through a maximum point $f'(x)$ is decreasing as x increases.

In Fig. 50 (a) the gradient of the tangent (1) is positive; that of (2) is positive and less than that of (1) and so on, that of (4) is zero, that of (5) is negative, that of (6) is negative and numerically greater than that of (5) and so on, so that all the way $f'(x)$ is algebraically decreasing.

Similarly from Fig. 50 (b) we see that as we pass through a minimum point $f'(x)$ increases as x increases, for we start with a large negative gradient $\searrow_1$ and gradually increase up to a large positive gradient $\nearrow^6$.

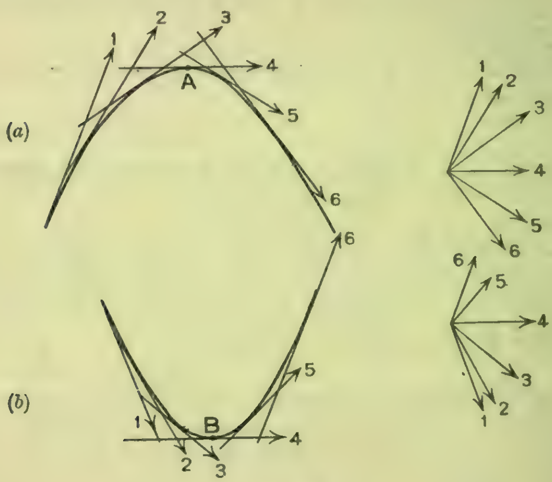


Fig. 50.

But by the above if $f'(x)$ increases as x increases, $f''(x)$ is positive, and if $f'(x)$ decreases as x increases $f''(x)$ is negative.

$\therefore$ If $f(x)$ is a maximum, $f'(x) = 0$ and **f″ (x) is negative.**

„ minimum, $f'(x) = 0$ and **f″ (x) is positive.**

We see then that where the curve is concave downwards $\frac{d^2y}{dx^2}$ is $-$, but where it is concave upwards $\frac{d^2y}{dx^2}$ is $+$. (See § 94.) Fig. 51. [Positive directions of axes as usual; v. p. 99.]

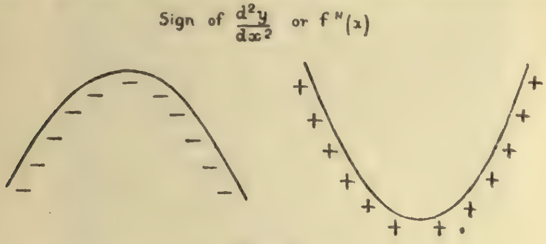


Fig. 51.

108. Consider the example in § 96.

$$f(x) = 2x^3 - 9x^2 + 12x - 3,$$
$$f'(x) = 6x^2 - 18x + 12$$
$$= 6(x^2 - 3x + 2),$$
$$f''(x) = 6(2x - 3).$$

The values of x which make $f'(x) = 0$ are 1 and 2.

If $\qquad x = 1, f''(x)$ is $-$.

If $\qquad x = 2, f''(x)$ is $+$.

$$\left.\begin{aligned} \therefore\ x = 1 &\text{ gives a maximum value of } f(x) \\ \text{and}\quad x = 2 &\text{ gives a minimum value of } f(x) \end{aligned}\right\}.$$

[See Fig. 41, p. 101.]

109. If we apply this test to the case in § 99,

where
$$f(x) = 3x^4 - 16x^3 + 30x^2 - 24x + 5,$$
$$f'(x) = 12(x^3 - 4x^2 + 5x - 2),$$
$$f''(x) = 12(3x^2 - 8x + 5).$$

The values of x which make $f'(x) = 0$ are 1 and 2.

If $x = 2$, $f''(x)$ is $+$, $\quad \therefore f(x)$ is a minimum,

but if $x = 1$, $f''(x) = 0$, and the test fails.

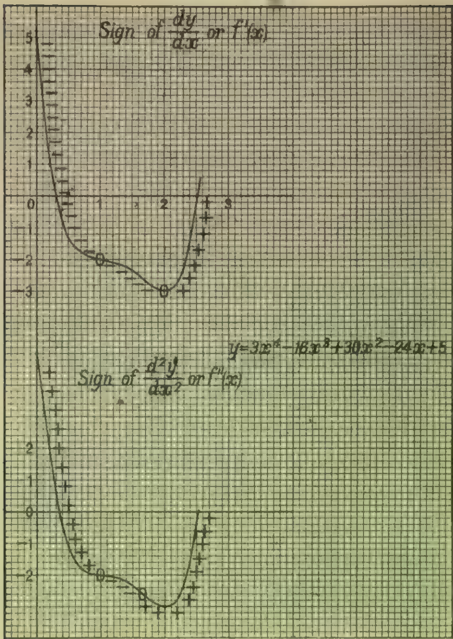


Fig. 52.

110. In the case of § 108, putting

$$f(x) = 2x^3 - 9x^2 + 12x - 3,$$

we have $$f'(x) = 6(x^2 - 3x + 2),$$

and $$f''(x) = 6(2x - 3).$$

If we draw the graphs of $y=f(x)$, $y=f'(x)$, and $y=f''(x)$ (Fig. 53) we get:

When $x=1$ or 2,

$f(x)$ is maximum or minimum,

$f'(x)=0$,

i.e. $y=f'(x)$ cuts the x-axis.

When $x = 1\frac{1}{2}$,

$$f'(x) \text{ is a minimum,}$$

$$f''(x) = 0,$$

and $$y = f''(x) \text{ cuts the } x\text{-axis.}$$

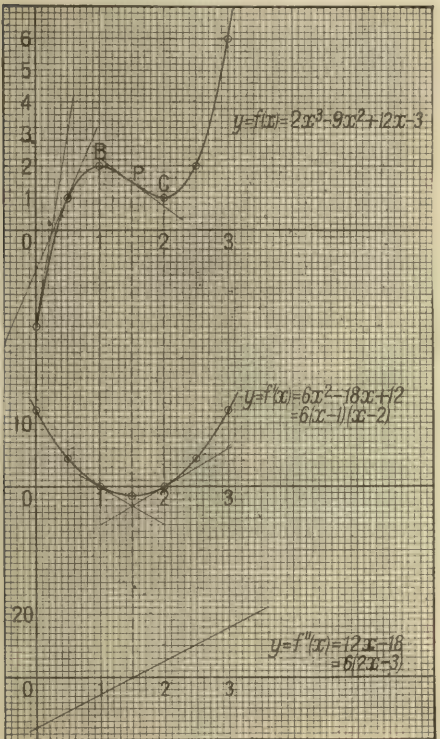


Fig. 53.

Points of inflexion.

111. A point like P where the gradient is a maximum or minimum is called a **point of inflexion.**

The portion BPC of the curve is shewn magnified in Fig. 54.

Between B and P the gradient diminishes (algebraically) steadily from 0 to $-1{\cdot}5$.

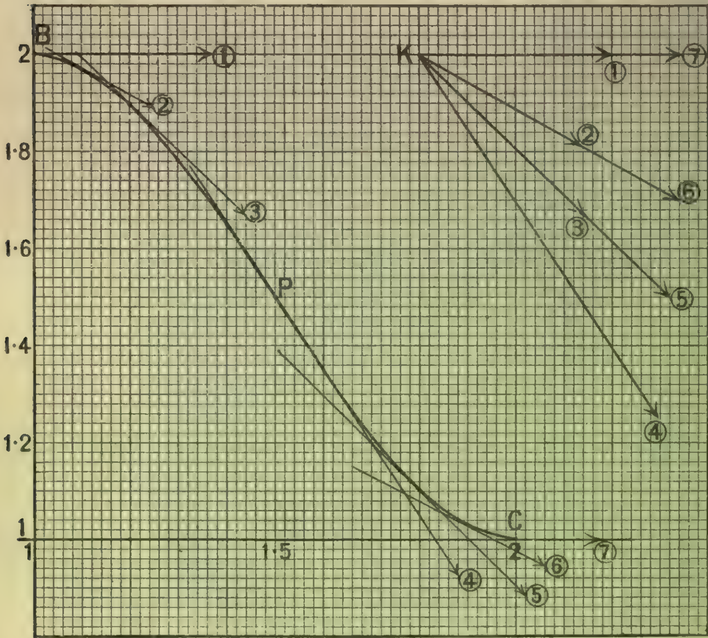


Fig. 54. Fig. 55.

Between P and C it increases (algebraically) from $-1{\cdot}5$ to 0.

The gradient is a minimum at P and the curve crosses the tangent there.

If we suppose a point to advance along the curve from B to C, and a line to be drawn from any point K parallel to the tangent at each position, this line swings in a clockwise direction until the moving point reaches P, and then back again in a counter-clockwise direction. (Fig. 55.)

The tangent at a point like P is sometimes called a **stationary tangent.**

Notice (v. Fig. 41, p. 101) that from A to P the curve is concave downwards and $\frac{d^2y}{dx^2}$ is $-$; from P to D the curve is concave upwards and $\frac{d^2y}{dx^2}$ is $+$.

At P where we change from one kind of bending to the other

$$\frac{d^2y}{dx^2}=0.$$

112. If we call a point where a curve *crosses* the x-axis a zero point we see

(1) **To a turning point on $y=f(x)$ corresponds a zero point on $y=f'(x)$.**

(2) **To a point of inflexion on $y=f(x)$ corresponds a turning point on $y=f'(x)$ and a zero point on $y=f''(x)$.**

113. In the case of § 109:

$$f(x)=3x^4-16x^3+30x^2-24x+5,$$
$$f'(x)=12(x^3-4x^2+5x-2)=12(x-1)^2(x-2),$$
$$f''(x)=12(3x^2-8x+5)=12(x-1)(3x-5).$$

When $x=2$, $f(x)$ is a minimum,

$f'(x)=0$,

$f''(x)$ is $+$.

When $x=1$,

$f(x)$ is neither a maximum nor a minimum [i.e. there is a point of inflexion],

$f'(x)=0$ and is a maximum,

$f''(x)=0$.

There is also another point of inflexion on $y=f(x)$ corresponding to $x=\frac{5}{3}$.

This makes $$f''(x) = 0,$$
$$f'(x) \text{ a minimum.}$$

114. Fig. 56 shews the graphs of

$$y = f(x),$$
$$y = f'(x),$$
$$y = f''(x).$$

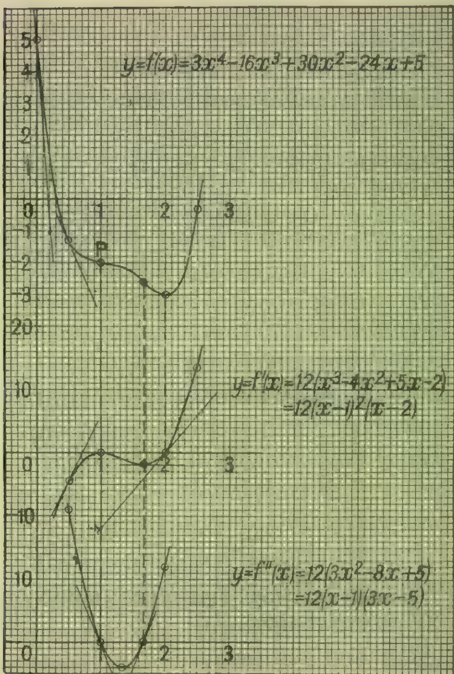


Fig. 56.

Notice when $x = 1$, $f'(x) = 0$, but $y = f'(x)$ does not cross the x-axis, and there is not a turning point on $y = f(x)$ corresponding

to this value of x. $f''(x)=0$, and $y=f''(x)$ crosses the x-axis, and we have a turning point on $y=f'(x)$ and a point of inflexion on $y=f(x)$.

115. We have seen that if a be a value of x which makes $f'(x)=0$, it may happen that $f(a)$ is not a maximum or minimum value of $f(x)$, and that usually if $f''(a)=0$, it will not, because then as a rule $f'(x)$ does not change sign as x passes through the value a.

It may happen, however, that even if $f''(a)=0$, $f'(x)$ changes sign as x passes through the value a, in which case we get a turning point corresponding to $x=a$.

Similarly if b be a value of x which makes $f''(x)=0$, it may happen that $f'(b)$ is not a max. or a min. value of $f'(x)$ and that usually if $f'''(b)=0$, it will not, because then, as a rule $f''(x)$ does not change sign as x passes through the value b.

116. Summary. If a is a value of x which makes $f'(x)=0$ and if $f''(a)\neq 0$, there is a turning point corresponding to $x=a$ and the value of $f(x)$ is a max. or min. according as $f''(x)$ is $-$ or $+$.

If $f'(a)=0$ and also $f''(a)=0$ we cannot tell without further investigation whether there is a turning point or not. If $f'(x)$ changes sign as x passes through the value a, there will be a turning point and $f(x)$ will be a max. or min. according as $f'(x)$ changes from $+$ to $-$ or from $-$ to $+$.

If b be a value of x which makes $f''(x)=0$ and if $f'''(b)\neq 0$ there is a point of inflexion corresponding to $x=b$. If $f''(b)=0$ and also $f'''(b)=0$ we cannot tell without further investigation whether there is a point of inflexion or not. If $f''(x)$ changes sign as x passes through the value b, there is a point of inflexion.

e.g. if
$$y=x^4(x+1),$$
we have
$$f(x)=x^5+x^4,$$
$$f'(x)=5x^4+4x^3,$$
$$f''(x)=20x^3+12x^2.$$

Now $f'(x)=0$, when $x=0$ or $-\cdot 8$.

When $x=-\cdot 8$, $f''(x)$ is $-$.

$\therefore$ $x=-\cdot 8$ gives a maximum value of $f(x)$.

When $x=0$, $f''(x)=0$.

But $$f'(x)=x^3(5x+4).$$

$$\therefore \left.\begin{array}{l}\text{If } x=0-,\ f'(x)=(-)(+)=-\\ \text{If } x=0+,\ f'(x)=(+)(+)=+\end{array}\right\}.$$

$\therefore$ $x=0$ gives a minimum value of $f(x)$.

Again $$f''(x)=4x^2(5x+3).$$

$$\therefore\ f''(x)=0 \text{ when } x=0 \text{ or } -\cdot 6,$$

$$f'''(x)=60x^2+24x=12x(5x+2).$$

When $$x=-\cdot 6,\ f'''(x)=(-)(-)=+.$$

$\therefore$ there is a point of inflexion when $x=-\cdot 6$ and the gradient is a minimum.

[Of course we could have seen this without using $f'''(x)$, for

if $$x=-\cdot 6-,\ f''(x)=(+)(-)=-;$$

if $$x=-\cdot 6+,\ f''(x)=(+)(+)=+.$$

$\therefore$ $f'(x)$ is a minimum and there is a point of inflexion.]

When $$x=0,\ f'''(x)=0.$$

In this case, if $x=0-$, $f''(x)=(+)(+)=+$;

if $x=0+$, $f''(x)=(+)(+)=+$.

$\therefore$ there is not a point of inflexion. [See Fig. 57.]

117. If we go on and calculate $f''''(x)$, etc. we get

$$f''''(x)=120x+24=+24 \text{ when } x=0,$$

and generally, it will be found that if $f'(x)=0$ when $x=c$, and if the first of the derived functions of x which does not vanish when $x=c$ is an even one, i.e. if it is one of the set $f''(x)$, $f^{\text{iv}}(x)$ etc., $x=c$ gives a turning point, but if the first one which does not vanish when $x=c$ is an odd one, i.e. if it is one of the set $f'''(x)$, $f^{\text{v}}(x)$ etc., $x=c$ does not give a turning point.

For instance in the above $f''''(x)$ was the first derived function which did not vanish when $x=0$. For a proof, the reader must refer to more advanced text-books.

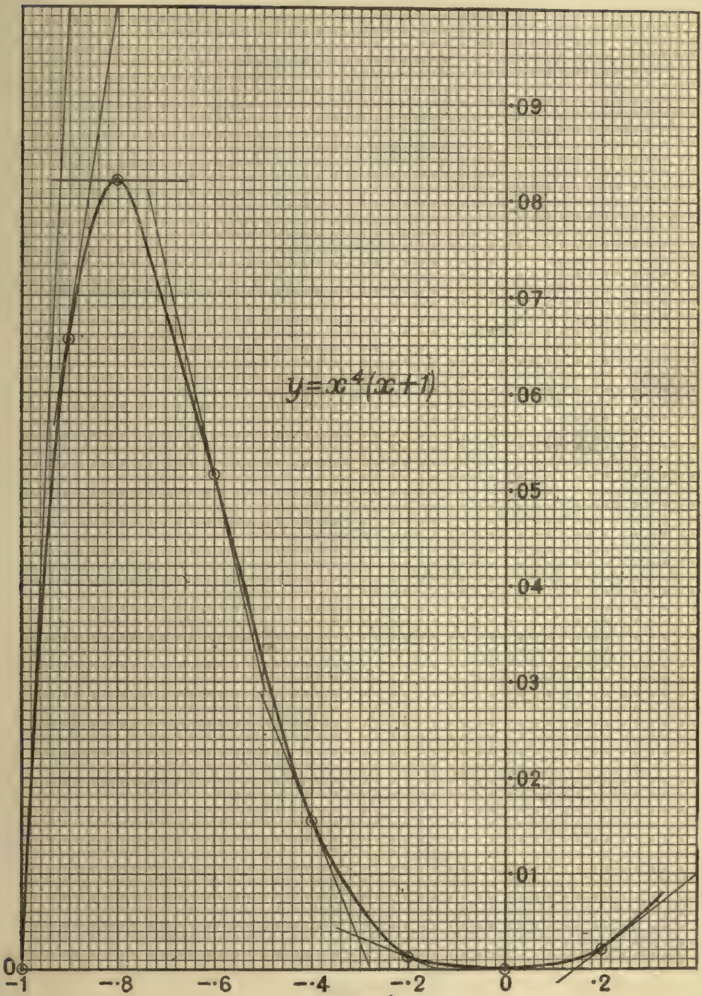


Fig. 57.

EXERCISES. XXV.

1. Find the maximum and minimum values of

$$2x^3 - 3x^2 - 36x + 10.$$

Also find the point of inflexion on

$$y = 2x^3 - 3x^2 - 36x + 10$$

and shew that it is the mid-point of the line joining the turning points.

2. Find the maximum and minimum values of

$$33x^2 - 4x^3 - 72x + 60.$$

3. In the curve $y = x^4 - 2x^3$, shew that there is a minimum point where $x = \frac{3}{2}$ and a point of inflexion where $x = 0$.

Find the co-ordinates of the other point of inflexion.

Draw the graph between $x = -1$ and $x = 3$.

4. Find a minimum value of $\frac{x^2}{3} + 2 + \frac{5}{x^2}$ and shew that there is no maximum.

5. Shew that there are two turning points on

$$y = x^6 (x - 2).$$

How many points of inflexion are there? Draw a sketch between $x = -1$ and $x = 2$.

6. If $\frac{dy}{dx} = (x+1)^3 (x-2)^4$ shew that $x = -1$ gives a minimum value of y, but that $x = 2$ gives neither a max. nor a min.

7. If $$f(x) = x^4 - 8x^3 + 24x^2 - 32x$$
shew that $f'(2) = 0$ and $f''(2) = 0$.

Is the point corresponding to $x = 2$ a turning point or a point of inflexion? Draw a rough sketch of $y = f(x)$ from $x = -1$ to $x = 5$.

8. In the curve

$$y = 2x^3 - 6x^2 - 18x + 7,$$

find the turning points and the point of inflexion and shew that the point of inflexion is midway between the turning points.

If the point of inflexion be taken as origin, the axes being parallel to the original axes, shew that the equation of the curve is

$$y = 2x^3 - 24x,$$

and that the curve is symmetrical in opposite quadrants.

9. Shew that $$y = ax^3 + bx^2 + cx + d$$
has always one point of inflexion and find its co-ordinates.

Shew also that if the curve has two turning points, the point of inflexion is midway between them.

Shew that the equation of the curve referred to parallel axes through the point of inflexion is
$$y = ax^3 + \left(c - \frac{b^2}{3a}\right)x,$$
and that the curve is symmetrical in opposite quadrants.

10. Shew that $$y = ax^4 + bx^3 + cx^2 + dx + e$$
has two points of inflexion or none according as $3b^2 - 8ac$ is + or −.

11. Draw on a large scale the graph of $y = 2x^4 - 3x^3 + 2x$ between $x = 0$ and $x = 1$.

CHAPTER V

SMALL ERRORS AND APPROXIMATIONS

118. We have already had a few examples of the application of the Differential Calculus to this kind of problem:

y being a given function of x, what will be the approximate change in y, due to a small change in x?

We make use of the fact that $\frac{\Delta y}{\Delta x}$ is approximately equal to $\frac{dy}{dx}$, the approximation being better as Δx is diminished. The approximation may be put in the form

$$\mathbf{\Delta y = \frac{dy}{dx} . \Delta x} \textbf{ approximately.}$$

e.g. if A sq. ins. is the area of a circle of radius r ins.

$$A = \pi r^2.$$

$$\therefore \frac{dA}{dr} = 2\pi r.$$

Now suppose a small error, which we may call Δr, has been made in measuring r, there will be a corresponding small error in A, which we may call ΔA, and we have seen that

$$\frac{\Delta A}{\Delta r} = \frac{dA}{dr} \text{ approximately,}$$

i.e. $$\Delta A = 2\pi r . \Delta r \text{ approximately.}$$

Suppose the radius was measured as 10 inches and that there was an error of ·1 inch.

Then we have $$r = 10, \ \Delta r = \cdot 1.$$

$$\therefore \Delta A = 2\pi \times 10 \times \cdot 1 \text{ approximately}$$
$$= 2\pi \text{ approximately,}$$

i.e. the error in the area is approximately 6·28 sq. ins.

[Shew that actually the error is 6·31 sq. ins.]

If the error in the radius was ·01″ we should get for the error in the area ·6284 sq. ins. instead of ·6287.

119. The result $\Delta A = 2\pi r \,.\, \Delta r$ approximately, admits of a simple geometrical interpretation which is important. It tells us that if the radius (r) of a circle be increased or decreased by a small amount (Δr) the increase or decrease in area is approximately $2\pi r \,.\, \Delta r$ or

Area of thin circular ring is approximately circumference of either boundary × breadth.

e.g. Suppose we have a ring with inner radius 20″ and breadth ·06″, the area is approximately

$$2\pi \times 20 \times \cdot 06 = 2 \cdot 4\pi = 7 \cdot 54 \text{ sq. ins. approximately.}$$

[Shew that area is actually $2 \cdot 4036\pi = 7 \cdot 55$ sq. ins.]

120. We might get this result from first principles thus.

Area of ring, inner and outer radii a and b,

$$= \pi (b^2 - a^2)$$

$$= \pi (b + a)(b - a) = \left(2\pi \,.\, \frac{b + a}{2}\right) . \, t \text{ where } t \text{ is the breadth}$$

= Circumference of concentric circle lying midway between the two circumferences × breadth. [Fig. 58.]

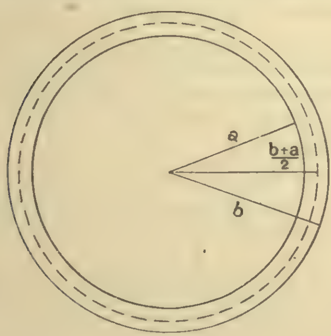


Fig. 58.

As the breadth is made less, the circumference of this intermediate circle comes nearer to the circumference of either boundary and can be brought as near to it as we please by making the breadth small enough.

EXERCISES. XXVI.

1. The radius (r) of a sphere is increased by a small length Δr. Shew that the increase in volume is approximately $4\pi r^2 . \Delta r$.

State this as a formula for the approximate volume of a thin spherical shell.

2. The radius of a sphere is 6″. Find approximately the diminution in volume if the radius is diminished by $\frac{1}{10}$″.

3. Find approximately the volume of a thin spherical shell, internal radius 1 foot, thickness ·2″. [Shew that your result is less than 2°/$_\circ$ in error.]

4. A stone is thrown at an angle of 40° to the horizontal with a velocity of 80 ft./sec. Find the range $\left[R = \frac{V^2 \sin 2\theta}{g}\right]$ and the approximate increase in the range if the velocity is changed to 82 ft./sec., the angle of projection remaining unchanged.

5. A cylindrical well is said to be 25 feet deep and 6 feet in diameter. Find the error in the calculated volume if there is an error of (i) 1″ in the diameter, (ii) 3″ in the depth.

6. The radius of the base of a cone is r and its vertical angle $2a$. Find the approximate increase in volume due to a small increase Δr in the radius, the vertical angle remaining constant. Hence shew that the volume of a conical shell, internal radius r, thickness t, t being small, is approximately πrlt where l is the slant height.

7. The radius of a sphere is found by measurement to be 18·5″, with a possible error of ·1″. Find the consequent errors possible in (i) the surface area, (ii) the volume, as calculated from this measurement.

8. $R = R_0(1 + at + bt^2)$ is a formula for the electrical resistance of a metal, R_0 being the resistance at 0° C. and t° C. the temperature.

Find the rate of increase of R per unit increase of t.

If $R_0 = 1·6$, $a = ·00388$, $b = ·000000587$, find the approximate change in R when the temperature rises from 100° to 101°.

121. Since $$\mathsf{A} = \pi r^2$$
and $$\Delta\mathsf{A} = 2\pi r \,.\, \Delta r \text{ approximately,}$$
$$\therefore \frac{\Delta\mathsf{A}}{\mathsf{A}} = 2 \,.\, \frac{\Delta r}{r}.$$

$\frac{\Delta\mathsf{A}}{\mathsf{A}}$ or the ratio of the error in the area to the original area, is called the **relative error** in the area.

Thus the relative error in the area is approximately twice the relative error in the radius.

If, for example, the radius increases by 1 °/₀, the area will increase approximately by 2 °/₀,

$$\text{for } \frac{\Delta r}{r} = \frac{1}{100}, \quad \therefore \frac{\Delta\mathsf{A}}{\mathsf{A}} = \frac{2}{100} \text{ approximately.}$$

EXERCISES. XXVII.

1. If V c. ins. is the volume of a cube of edge x ins. prove
$$\frac{\Delta V}{V} = 3 \,.\, \frac{\Delta x}{x} \text{ approximately.}$$
Hence find approximately the percentage increase in volume due to an increase of ·5 °/₀ in the edge.

2. Prove that the relative increase in the volume of a sphere is approximately 3 times the relative increase in the radius.

3. If $pv = k$, prove $\frac{\Delta p}{p} = -\frac{\Delta v}{v}$ approximately.

If the pressure increase 1 °/₀, what is the approximate change in the volume?

4. If $pv^{1\cdot4} = k$, prove $\frac{\Delta p}{p} = -1\cdot4\,\frac{\Delta v}{v}$ approximately.

5. If $y = x^n$, prove $\frac{\Delta y}{y} = n \,.\, \frac{\Delta x}{x}$ approximately.

122. Elasticity of volume. Suppose v c. ins. to be the volume of unit mass of a fluid and p lbs./sq. in. the pressure, p being some given function of v.

If p be increased to $p + \Delta p$, v will become $v + \Delta v$ (Δv being,

of course, negative, since an increase in pressure will produce a decrease in volume).

$-\Delta v$ will be the diminution of volume.

$-\dfrac{\Delta v}{v}$ (the ratio of this diminution to the original volume) is called the **volume strain.**

The ratio of Δp (the increase of pressure required to produce this) to the volume strain is $-v\,.\,\dfrac{\Delta p}{\Delta v}$ and the limit of this when Δv is indefinitely diminished, i.e. $-v\dfrac{dp}{dv}$, is called the **elasticity of volume** or the bulk modulus of the fluid.

6. If $pv=$constant, prove that the elasticity of volume is p.

7. If $pv^n=$constant, prove that the elasticity of volume is np.

8. If p be given as a function of v, say $p=f(v)$ and the graph $p=f(v)$ be drawn, the v-axis being horizontal, shew that if P be a point on the graph corresponding to any given value of v and MQ be drawn through the foot of the ordinate of P parallel to the tangent at P, meeting the p-axis in Q, then OQ will give the elasticity of volume for this value of v. [Fig. 59.]

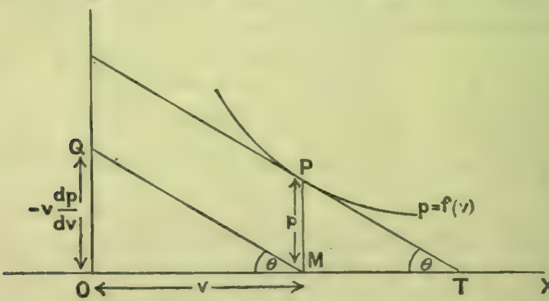


Fig. 59.

9. A formula giving the deflection (D) of a beam in terms of the length (l), the load (W), the Moment of Inertia of the Cross-Section (I) and Young's modulus (E) is $D=\dfrac{Wl^3}{EI}\times\dfrac{1}{48}$.

Shew that if a small error be made in l, the resulting percentage error in D is approximately 3 times the percentage error in l.

10. The velocity of discharge (v ft./sec.) of water through a long pipe is given by $v^{1\cdot 87}=\frac{H}{\cdot 0004L}\times d^{1\cdot 4}$ where H ft. = head of water, d ft. = diameter of pipe, L ft. = length of pipe.

Find the velocity of discharge when the head is 35 feet, the length of pipe 1 mile and its diameter 18 inches.

If a small error Δd is made in estimating the diameter, prove that the error (Δv) in the velocity $=\frac{1\cdot 4}{1\cdot 87}\times\frac{v}{d}\times\Delta d$ approximately.

If the error in the diameter is $\frac{1}{10}''$, find Δv.

11. If t secs. is the time of oscillation of a pendulum of length l feet

$$t=2\pi\sqrt{\frac{l}{g}}. \quad [g=32\cdot 2.]$$

Shew that the relative error in t is approximately half that in l.

Calculate t when $l=4$ and find approximately the error in t corresponding to an error of $\frac{1}{2}''$ in the length.

12. If $pv^{1\cdot 8}=k$ and the pressure increase $2\,^\circ\!/_\circ$, what is approximately the percentage change in volume?

123. $$\frac{\Delta y}{\Delta x}=\frac{dy}{dx} \text{ approximately,}$$

i.e. if x is increased by Δx, y is increased by

$$\Delta x\times\frac{dy}{dx} \text{ approximately.}$$

Suppose $y=f(x)$, then $\frac{dy}{dx}=f'(x)$, and the above statement is equivalent to this:

If x is increased by a small quantity h, then $f(x)$ is increased by $hf'(x)$ approximately. [h taking the place of Δx and $f'(x)$ of $\frac{dy}{dx}$.] Or

$$\mathbf{f(x+h)=f(x)+hf'(x)} \textbf{ approximately.}$$

124. Let P be the point (x, y) on the curve $y = f(x)$ and let Q be a neighbouring point whose abscissa ON is $x + h$. [Fig. 60.]

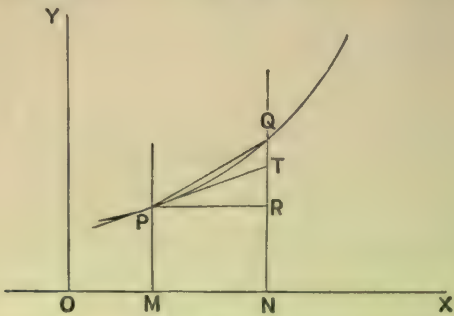


Fig. 60.

Then
$$NQ = f(x+h),$$
$$MP = f(x).$$

$\therefore$ if PR is parallel to OX, $RQ = f(x+h) - f(x)$.

If PT is the tangent at P, the gradient of PT is $f'(x)$ and
$$RT = hf'(x),$$
so that in taking $f(x+h) - f(x)$ as equal to $hf'(x)$ we take PQ as coincident with PT, in other words we consider the portion PQ of the curve as straight.

125. *Ex.* Find the value of
$$3x^5 - 4x^3 + 6x^2 - 7x + 8 \quad \text{when} \quad x = 3\cdot01.$$

Calling the expression $f(x)$ we have
$$f'(x) = 15x^4 - 12x^2 + 12x - 7.$$
$$\therefore f(3) = 662, \; f'(3) = 1136.$$
$$\therefore f(3\cdot01) = 662 + 1136 \times \cdot01 \text{ approximately}$$
$$= 673\cdot36 \text{ approximately}.$$

126. Notice that if $f'(x) = 0$, i.e. if the tangent at P is parallel to OX, this approximation will not shew any difference between $f(x)$ and $f(x+h)$. See Exs. 5 and 6 below.

EXERCISES. XXVIII.

1. Shew that if $f(x)=ax+b$ where a and b are constants the statement

$$f(x+h)=f(x)+hf'(x)$$

is accurately true for all values of h.

2. Find approximately the value of $5x^2+4x+9$ when $x=2{\cdot}0087$.

3. Find approximately the value of $6x^3-7x^2+2x-9$ when $x=3{\cdot}002$.

4. Find approximately the value of $7x^2-3x+\frac{4}{x}$ when $x=2{\cdot}01$.

5. Find approximately the value of

$$2x^3-9x^2+12x-3 \text{ when } x=\text{(i) } 2{\cdot}005, \text{ (ii) } 1{\cdot}995.$$

In each case find also the accurate value.

6. Find approximately the value of

$$3x^4-16x^3+30x^2-24x+5 \text{ when } x=\text{(i) } 2{\cdot}01, \text{ (ii) } 1{\cdot}01.$$

In each case find the accurate value.

127. The approximation $f(x+h)=f(x)+hf'(x)$ is, as we have seen, equivalent to the statement that the gradient of PQ [Fig. 61] is that of the tangent at P. If we suppose P and Q to be such that $f'(x)$ and $f''(x)$ do not change sign between P and Q, so that between these points the curve has one of the shapes shewn on p. 99, it is clear that the gradient of PQ is between the gradients of the tangents at P and Q and equal to the gradient at some intermediate point. Thus the quantity [represented by RQ] added to $f(x)$ to make $f(x+h)$ is between $hf'(x)$ [RT] and $hf'(x+h)$ [RU] and is actually $hf'(x+\theta h)$ where θ is some proper fraction, so that $(x+\theta h)$ is between x and $(x+h)$.

e.g. in § 125 $f'(3)=1136$, $f'(3{\cdot}01)=1151{\cdot}68$.

So that

$$f(3{\cdot}01)-f(3) \text{ is between } {\cdot}01\times f'(3) \text{ and } {\cdot}01\times f'(3{\cdot}01),$$
$$\text{i.e. between } 11{\cdot}36 \text{ and } 11{\cdot}52.$$

Actually it is $11{\cdot}43826...$, and as a matter of fact

$$f(3{\cdot}005)=1143{\cdot}8200,$$

so that in this case $f(x+h)-f(x)=hf'(x+\tfrac{1}{2}h)$.

[Correct to 3 decimal places.]

The more rapidly $f'(x)$ is changing, the less reliable will $f(x)+hf'(x)$ be as an approximation to $f(x+h)$.

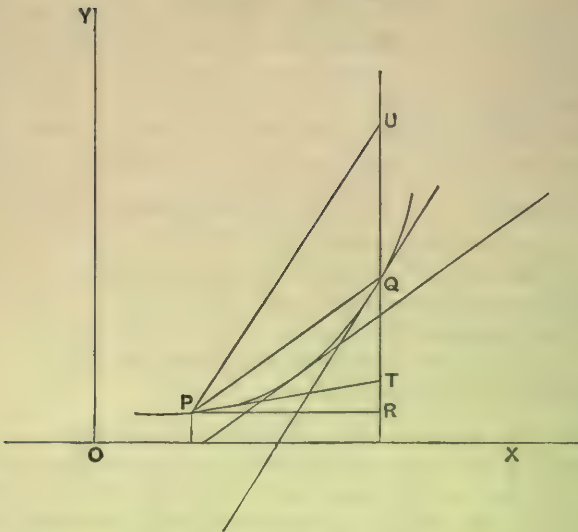


Fig. 61.

EXERCISES. XXIX.

1. If $f(x) \equiv ax^2 + bx + c$, shew that $f(x+h) = f(x) + hf'(x + \frac{1}{2}h)$.
In particular, if $f(x) \equiv 7x^2 - 11x + 15$, shew that $f(3{\cdot}4) = f(3) + {\cdot}4f'(3{\cdot}2)$.

2. If $f(x) \equiv x^3 - 6x - 5$, shew that

$$f(1{\cdot}4) - f(1) = -0{\cdot}656,$$
$${\cdot}4f'(1) = -1{\cdot}2,$$
$${\cdot}4f'(1{\cdot}4) = -0{\cdot}048,$$
$${\cdot}4f'(1{\cdot}205) = -0{\cdot}65757.$$

3. If $f(x) \equiv x^4 - 4x^3 + 16x - 7$, shew that

(i) $f(3{\cdot}2) - f(3) = 3{\cdot}9856,$
$${\cdot}2f'(3) = 3{\cdot}2,$$
$${\cdot}2f'(3{\cdot}2) = 4{\cdot}8384,$$
$${\cdot}2f'(3{\cdot}102) = 3{\cdot}9852.$$

(ii) $f(4) - f(3) = 43,$
$$f'(3) = 16,$$
$$f'(4) = 80,$$
$$f'(3{\cdot}539) = 43{\cdot}003.$$

CHAPTER VI

THE INVERSE OPERATION

128. GIVEN $\frac{dy}{dx}$, find y.

We know that if $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$.

Can we say that if $\frac{dy}{dx} = nx^{n-1}$ then $y = x^n$? Not unless we are certain that x^n is the *only* function whose differential coefficient is nx^{n-1}. As a matter of fact we have seen that if $y = x^n +$ any constant whatever, $\frac{dy}{dx} = nx^{n-1}$. $\therefore$ if we are given that $\frac{dy}{dx} = nx^{n-1}$, we can only conclude that $y = x^n +$ some constant, and if no further information is given "some constant" may be replaced by "any constant whatever." As a rule sufficient information is given to enable us to fix the constant for the particular case in question.

In practice $\frac{dy}{dx}$ will not present itself in such a convenient form as nx^{n-1}, e.g. we might have the value of $\frac{dy}{dx}$ given not as $5x^4$ but as x^4 or $3x^4$. To deal with such cases we have only to remember that if $y = k \,.\, x^n$ where k is a constant, $\frac{dy}{dx} = k \,.\, nx^{n-1}$. Thus $3x^4 = \frac{3}{5} \,.\, 5x^4$ and therefore arises from differentiating $\frac{3}{5} \,.\, x^5$.

Remembering what was said about the addition of an arbitrary constant, we see that if $\frac{dy}{dx}=3x^4$ then $y=\frac{3}{5}x^5+c$ where c is some constant.

Similarly if $\frac{dy}{dx}=5\sqrt{x}$, i.e. $5x^{\frac{1}{2}}$, we know that $x^{\frac{1}{2}}$ must come from differentiating $x^{\frac{1}{2}+1}$, i.e. $x^{\frac{3}{2}}$: but if $y=x^{\frac{3}{2}}$, $\frac{dy}{dx}=\frac{3}{2}x^{\frac{1}{2}}$.

$\therefore$ if $\frac{dy}{dx}=5x^{\frac{1}{2}}$, $y=5\times\frac{2}{3}x^{\frac{3}{2}}+c$.

EXERCISES. XXX.

1. Write down the values of y corresponding to the following values of $\frac{dy}{dx}$:

(i) $3x^2$. (ii) $3x^3$. (iii) $\frac{2}{x^2}$. (iv) 7. (v) $5x^{-\frac{1}{2}}$. (vi) $\frac{2}{3}x^n$.

(vii) $\frac{1}{2x^3}$. (viii) $\sqrt[3]{x}$. (ix) $\frac{1}{\sqrt[4]{x}}$. (x) $\frac{3}{5}x^{\frac{7}{4}}$. (xi) $2x^3+3x$. (xii) $\frac{3}{x^2}+4$.

(xiii) $7x^2+3x+\frac{2}{x^3}$. (xiv) $3(4x^3+2\sqrt{x})$.

2. What is $f(x)$ if $f'(x)=$

(i) x^2. (ii) $3x^2-x+1$. (iii) $\frac{5}{x^2}+3$?

129. If $\frac{d^2y}{dx^2}$ is given there will be two arbitrary constants in the value of y, as the following example will shew:

Given $\frac{d^2y}{dx^2}=2x+3$, find y.

Since $\frac{d^2y}{dx^2}$ means $\frac{dz}{dx}$ where z stands for $\frac{dy}{dx}$, we have

$$\frac{dz}{dx}=2x+3.$$

$$\therefore\ z=x^2+3x+a$$

where a is *any* constant.

i.e. $$\frac{dy}{dx} = x^2 + 3x + a.$$

$$\therefore\ y = \frac{x^3}{3} + \frac{3x^2}{2} + ax + b$$

where b is *any* constant.

EXERCISES. XXXI.

1. Find the values of y corresponding to the following values of $\frac{d^2y}{dx^2}$:

(i) $3x^2$. (ii) $\frac{5}{x^3}$. (iii) 7. (iv) $\frac{2x}{3}+5$. (v) $\frac{2}{3x^4}$. (vi) $\sqrt{x}$.

2. If $f''(x) = 3x - 8$, find $f(x)$.

3. If $f''(x) = 5$, find $f(x)$.

4. If $f''(x) = 2x^3 - 3x + 1$, find $f(x)$.

130. The following examples will shew how to fix the value of the arbitrary constant in special cases.

Ex. 1. Given that $\frac{dy}{dx} = 3x$, and that $y = 5$ when $x = 2$, find y in terms of x.

We have $$y = \frac{3}{2}x^2 + c.$$

c must be chosen so that this shall be satisfied when

$$x = 2 \text{ and } y = 5.$$

$$\therefore\ 5 = \frac{3}{2}.4 + c.$$

$$\therefore\ c = -1.$$

$$\therefore\ y = \frac{3}{2}x^2 - 1.$$

EXERCISES. XXXII.

1. Given $\dfrac{dy}{dx}=x^2+1$ and $y=3$ when $x=3$, express y in terms of x.

2. Given $\dfrac{dp}{dv}=v+\dfrac{1}{v^2}$ and $p=5$ when $v=2$, express p in terms of v.

3. Given $\dfrac{dy}{dx}=\sqrt{x}$ and $y=7$ when $x=4$, find y in terms of x.

4. Given $\dfrac{dA}{dy}=5y+4$ and $A=6$ when $y=1$, express A in terms of y.

5. Given $\dfrac{dy}{dx}=(x+1)(x+2)$ and $y=12$ when $x=3$, express y in terms of x.

6. Given $\dfrac{ds}{dt}=(2t-3)^2$ and $s=52$ when $t=5$, express s in terms of t.

7. Given $\dfrac{dV}{dr}=4\pi r^2$ and $V=36\pi$ when $r=3$, find V in terms of r.

8. Given $\dfrac{dy}{dx}=\dfrac{1}{\sqrt[3]{x}}$ and $y=8$ when $x=27$, find y in terms of x.

9. The speed of a body at the end of t secs. is given by the formula $v=u+at$ where u and a are constants. Find the relation between s and t, given $s=0$ when $t=0$.

10. Given $\dfrac{d^2y}{dx^2}=3x$; also $y=11$ when $x=2$, and $y=47$ when $x=4$, find y.

11. Given $\dfrac{d^2y}{dx^2}=32$; also $y=0$ when $x=0$, and $\dfrac{dy}{dx}=5$ when $x=0$, find y.

131. *Ex.* 2. The gradient of a curve at the point (x, y) is $2x+3$, and the curve passes through $(1, 2)$. Find its equation.

We have
$$\frac{dy}{dx}=2x+3.$$
$$\therefore\ y=x^2+3x+c$$
[where c is some constant to be fixed].

This has to be satisfied by $x = 1$, $y = 2$.

$$\therefore\ 2 = 1 + 3 + c.$$

$$\therefore\ c = -2.$$

$\therefore$ the equation of the curve is

$$\mathbf{y = x^2 + 3x - 2.}$$

132. If we had not been given the second piece of information we should have had the equation of the curve

$$y = x^2 + 3x + c$$

where c is any constant whatever.

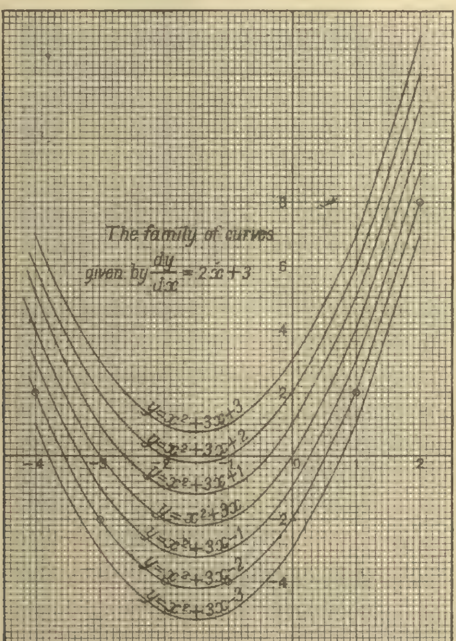


Fig. 62.

For different values of c this really represents a whole family of curves got by taking

$$y = x^2 + 3x$$

and sliding it through any distance parallel to Oy.

$$y = x^2 + 3x - 2$$

is that particular member of the family which goes through (1, 2). (Fig. 62.)

We might say that $\frac{dy}{dx} = 2x + 3$ represents a family of curves, viz. that obtained by drawing $y = x^2 + 3x + c$ for different values of c.

EXERCISES XXXIII.

1. Find the equation of a curve whose gradient at any point (x, y) is $3 - 4x$ and which passes through (2, 3).

2. Find the equations of the curves with the same gradient-law as in Ex. **1** passing through (i) (0, 0), (ii) $(-3, 1)$.

3. Draw the 3 curves of Exs. **1** and **2**.

4. The gradient of a certain curve at any point (x, y) is $2x + 1$, and the curve passes through the point (1, 5). Find the equation of the curve and the equation of the tangent at the point (1, 5).

5. The gradient of a certain curve at the point (x, y) is $2x^2 + 3x - 7$, and the curve passes through (1, 2). Find its equation and also the equation of the normal at the point (1, 2).

6. The gradient of a curve at the point (x, y) is $3x - 5$. Shew that it has a minimum point and find its co-ordinates (i) if the curve passes through the origin, (ii) if the curve passes through $(-3, 2)$.

7. Find the equation of a curve which passes through the point (3, 5) and whose gradient-law is $\frac{dy}{dx} = 7 - 4x + x^2$.

In what point and at what angle does this curve cut the y-axis?

133. *Ex.* 3. A point is moving in a straight line so that its speed at the end of t secs. is $(3t + 2t^2)$ ft./sec. and its distance from a fixed point in the line at the end of 2 seconds is 20 feet. Find its distance from the fixed point at the end of t secs.

If the required distance is s ft. we have

$$\frac{ds}{dt} = 3t + 2t^2.$$

$$\therefore \quad s = \frac{3}{2}t^2 + \frac{2}{3}t^3 + c$$

[where c is a constant to be determined].

Now we are told that when $t = 2$, $s = 20$.

$$\therefore \quad 20 = \frac{3}{2}\cdot 4 + \frac{2}{3}\cdot 8 + c.$$

$$\therefore \quad c = 8\tfrac{2}{3}.$$

$$\therefore \quad \mathbf{s = \frac{3}{2}t^2 + \frac{2}{3}t^3 + \frac{26}{3}}.$$

Ex. 4. A point is moving in a straight line so that its acceleration at the end of t seconds is $(3 + 4t)$ ft./sec.^{2} At the end of 2 seconds its speed is 16 ft. per second and its distance from a fixed point in the line is $20\frac{1}{3}$ feet. Find its distance from the fixed point at the end of t seconds.

Let the required distance be s ft. and let its speed at the end of t secs. be v ft./sec.

We have $$\frac{dv}{dt} = 3 + 4t.$$

$$\therefore \quad v = a + 3t + 2t^2,$$

where a is some constant to be determined.

Now when $t = 2$, $v = 16$.

$$\therefore \quad 16 = a + 6 + 8. \quad \therefore \ a = 2.$$

$$\therefore \quad v = 2 + 3t + 2t^2.$$

i.e. $$\frac{ds}{dt} = 2 + 3t + 2t^2.$$

$$\therefore \quad s = b + 2t + \frac{3}{2}t^2 + \frac{2}{3}t^3,$$

where b is some constant to be determined.

Now when $t=2$, $s=20\frac{1}{3}$.

$$\therefore\ 20\tfrac{1}{3}=b+4+6+5\tfrac{1}{3}. \quad \therefore\ b=5.$$

$$\therefore\ s=5+2t+\frac{3}{2}t^2+\frac{2}{3}t^3.$$

EXERCISES. XXXIV.

1. The speed of a body moving in a straight line is given by

$$v=60-32t.$$

Its distance from a fixed point O in the line at the end of 1 second is 44 feet, find its distance from O at the end of t seconds.

2. The acceleration of a body moving in a straight line is 32 ft./sec.^{2} Its speed at the end of 2 seconds is 154 ft. sec. and its distance from a fixed point O in the line is 234 feet. Find its speed and distance from O at the end of 3 seconds.

3. The acceleration of a body moving in a straight line at the end of t seconds is $(2+3t)$ ft./sec.^{2} Its distances from a fixed point O in the line at the end of 1 and 2 seconds are respectively $7\frac{1}{2}$ ft. and 15 ft.; find s in terms of t and get the distance from O and the speed of the body (i) when $t=0$, (ii) when $t=5$.

4. A body moves in a straight line with constant acceleration a ft./sec.^{2} Its initial speed is u ft./sec. If v ft./sec. is its speed at the end of t secs., and s ft. its distance from the initial position at the end of t secs., prove

$$v=u+at \quad \text{and} \quad s=ut+\frac{1}{2}at^2.$$

5. If $\dfrac{dp}{dv}=-\dfrac{700}{v^{2\cdot4}}$ express p as a function of v, given that $p=18{\cdot}95$ when $v=20$.

6. If $\dfrac{dy}{dx}=ax+b$ express y as a function of x, given the following pairs of corresponding values of x and y:

x	0	1	2
y	3	5	9.

MISCELLANEOUS EXAMPLES ON CHAPTERS I—VI.

A.

1. From first principles find $\frac{dy}{dx}$ when

$$y=3x^3+2x+1.$$

Also write down the equation of the tangent to the curve represented by $y=3x^3+2x+1$ at the point where $x=1$.

2. A body starts from rest and moves in a straight line in such a way that its speed after t seconds is $(16t-4t^2)$ ft./sec. Find the distance described during the fourth second.

3. The strength of a rectangular beam varies as bd^2 where b is the breadth and d the depth. Find the depth and breadth of the strongest rectangular beam of perimeter 3 feet.

4. The radius of a spherical bubble grows at the rate of ·01 inch per second. At what rate is (i) the surface, (ii) the volume increasing when the radius is (a) 1 inch, (b) 1 foot?

5. The volume of a cylindrical tin canister closed at both ends is 300 c.c. Find the most economical dimensions.

B.

1. (i) Find from first principles $\frac{dy}{dx}$ when $y=\frac{1}{x}$.

(ii) Find the equations of the tangent and normal to the curve $xy=1$ at the point P, where $x=5$.

(iii) If the tangent meets the axis in T and t, shew that $\mathrm{PT}=\mathrm{P}t$.

2. Write down (i) $\frac{dy}{dx}$ when $y=5\sqrt{x}+\frac{2}{\sqrt{x}}$.

(ii) $\frac{dV}{dx}$ if $V=\frac{\pi x^2}{3}(R-x)$, where R is constant.

3. Water is poured at the rate of 5 cubic feet per minute into a vessel in the shape of a hollow cone with a vertical angle of 90°. When the vessel contains 12 c. ft., at what rate is the depth increasing?

4. The sum of the perimeters of two equal squares and a circle is 100 feet. When is the sum of the areas least, and when is it greatest?

Are these really maximum and minimum values?

5. There is a certain curve such that the gradient at any point (x, y) is $3x$. Find its equation, given that it passes through the origin.

C.

1. If $y=3x^2+2$, obtain from first principles the differential coefficient of y with respect to x, and shew with a diagram the geometrical meaning of your result. For what value of y is this differential coefficient equal to unity?

2. If $$y=2x^3-27x^2-132x+2$$ find its maximum and minimum values.

3. A lamp is 60 feet above the ground. A stone is let drop from a point at the same level as the lamp and 20 feet away from it. Find the speed of the shadow of the stone on the ground (i) after 1 second, (ii) when it has fallen 30 feet. [$g=32$.]

4. Find the co-ordinates of the point of intersection of the tangents to $y=3x^2+5x-7$ at the points where $x=2$ and $x=5$, and the equation of the line joining this point of intersection to the mid-point of the chord joining the points of contact.

5. The graph of $f'(x)$ is the parabola
$$y+2=3(x-1)^2.$$
What information does this give respecting $j(x)$?

What is the gradient at the point on $y=f(x)$ where $x=1$?

D.

1. A body is thrown into the air with a velocity of 100 ft./sec. at a certain elevation. If the resistance of the air be neglected, the horizontal distance from the point of projection (x feet) and the height above the ground (y feet) after t seconds are given by the equations $x=80t$, $y=60t-16t^2$. Find the horizontal and vertical velocities after 3 seconds, and the magnitude and the direction of their resultant. By eliminating t between the given equations, get the equation of the path in the form $y=\frac{3}{4}x-\frac{1}{400}x^2$, and find the direction of the tangent to this path when $x=240$.

2. Find the maximum and minimum values of
$$(x+3)(x-1)(x-3).$$

3. Find the equations of the tangents to the curve

$$5y=(x+3)(x-1)(x-3)$$

at the points where the curve crosses the axes and also the co-ordinates of the point of inflexion.

4. The volume of a sphere is V cubic inches and the surface S square inches. Shew that $V=\frac{1}{6\sqrt{\pi}}\cdot S^{\frac{3}{2}}$.

The surface of a sphere is given as 100 square inches, and from this the volume is calculated. If the surface is really 101 square inches, find the approximate error in the calculated volume.

5. The gradient of a certain curve obeys the law $\frac{dy}{dx}=4x^2+3$ and the curve goes through the point (1, 1). Find its equation.

E.

1. The surface of a sphere is given and the volume calculated from it. If there is an error of $\cdot 4\ {}^{\circ}\!/_{\circ}$ in the surface, find approximately the resulting percentage error in the calculated volume.

2. An open tank with a square base and vertical sides is to have a capacity of 4000 cubic feet. Find the dimensions so that the cost of lining it with lead may be a minimum.

3. The equation of the path of a projectile thrown in vacuo with a velocity of 64 ft./sec. at an angle of 45° to the horizon is $y=x-\frac{x^2}{128}$, the origin being the point of projection, the x-axis horizontal, and the unit along each axis 1 foot.

Find:

(i) The co-ordinates of the highest point.

(ii) The equation of the tangent at the point for which $x=32$.

(iii) Where the direction of motion makes an angle 30° with the horizon.

4. If $y=2x^4-2x^3-x^2+1$, find y when $x=-1, -\frac{1}{4}, 0, 1, 2.$

Also find $\frac{dy}{dx}$ and the values of $\frac{dy}{dx}$ when $x=-1, 2.$

Also find what values of x make $\frac{dy}{dx}=0.$

Using all this information, draw the graph of

$$y=2x^4-2x^3-x^2+1 \text{ between } x=-1 \text{ and } x=2.$$

$$\left[x\text{-scale unit } 2'', \quad y\text{-scale unit } \frac{1''}{2}.\right]$$

5. In a certain case of straight line motion, the velocity in feet per second, at a given instant t seconds after the start, is given by the formula

$$v=3t+4t^2.$$

Calculate:

(i) The average velocity between the end of 2 seconds and the end of 4 seconds.

(ii) The arithmetic mean of the velocities at the instants 2 and 4 seconds after the start.

(iii) The velocity 3 seconds after the start.

F.

1. If v_0 be the volume of a given mass of water at 0° C., and v its volume at θ° C., Hällström's formula is

$$v=v_0(1-a\theta+b\theta^2-c\theta^3)$$

for values of θ between 0 and 30, where

$$a=\cdot 000057577,$$
$$b=\cdot 0000075601,$$
$$c=\cdot 00000003509.$$

What is the temperature of maximum density?

2. Draw the graph of $y^2=x(x-1)^2$ between $x=0$ and $x=3$. Find the co-ordinates of the turning points and the equations of the tangents at (1, 0).

3. The normal at a point of $y^3=x$ meets OX in G. Prove that the minimum value of OG is ·7698.

4. A body of mass 20 lbs. is moving in a straight line so that

$$s=3-5t+4t^3,$$

s feet being its displacement from some standard position at the end of t seconds. Find its kinetic energy and momentum and the force acting on it at the end of 4 seconds.

5. Find approximately the value of $16-2\cdot 5x+4\cdot 7x^2$, when $x=\cdot 997$.

G.

1. In a certain case of straight line motion the acceleration is proportional to the time which has elapsed since the moving body passed a fixed point A. Find the distance of the body from A after 10 seconds, given that its speed at A is 8 ft./sec. and its distance from A after 1 second is 9 ft.

2. If $y=ax^2+bx$ where a and b are constants, prove

$$x^2 . \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

3. A piece of wire, a inches long, is cut into two parts, one of which is bent into the form of a square and the other into that of a circle. Find where it must be cut so that the sum of the areas enclosed may be a minimum and shew that the diameter of the circle is equal to the side of the square.

4. P is the point on the curve $y=x^3$ whose co-ordinates are (h, h^3). The tangent and normal to the curve at P meet the axis of y in T and G respectively, and PN is the perpendicular from P on the y axis. Prove that OT=2ON, and that as P changes its position on the curve, NG varies inversely as NP.

5. The pressure and volume of a gas are connected by the relation $pv^{\cdot 8}=$constant. When the volume is 20 cubic feet the pressure is 30 lbs. per sq. ft. Find the elasticity of volume when the volume is 40 cubic feet.

H.

1. If $f(x)=(x+1)(x-2)^3$, find $f'(x)$, $f''(x)$, and $f'''(x)$.

Find the values of x which make

(i) $f(x)=0$, (ii) $f'(x)=0$, (iii) $f''(x)=0$, (iv) $f'''(x)=0$.

Find the co-ordinates of the maximum and minimum points on

$$y=f(x), \quad y=f'(x), \quad y=f''(x),$$

and the points of inflexion on $y=f(x)$ and $y=f'(x)$.

Also calculate the values of $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$, when $x=-1$, 0, 1, 2, 3.

Using the information you have now collected draw on squared paper the curves

$$y=f(x), \quad y=f'(x), \quad y=f''(x), \quad y=f'''(x),$$

using the same x-scale for all.

2. A conical vessel has vertical angle 60°. Water is flowing into it at the rate of 1 cubic foot per minute. Shew that when the water is 1 foot deep the level is rising at about ·19 inch per second.

3. The tangent at P to the curve $ay^2=x^3$ meets OX in T and OY in t. PN is the ordinate of P.

Prove $\mathrm{OT}=\frac{1}{3}\,\mathrm{ON}$ and $\mathrm{O}t=\frac{1}{2}\,\mathrm{PN}$.

4. A solid sphere radius r floats just immersed in water. Find where the sphere must be cut by a horizontal plane, so that if the top part be removed the thrust on the plane surface of the remainder may be a maximum.

5. Draw the circle $x^2+y^2-6x-8y+21=0$, and by drawing find the points on it at which $\frac{dy}{dx}=\frac{y}{x}$.

I.

1. A vertical line moves with uniform speed of 1″ per second, so that its lower end always lies in the axis and its upper end on the parabola $y=x^2$. [The unit being 1″ each way.] At what rate is the length of the line growing when it is 10″ from the origin?

2. P, Q are any two points on the parabola $y=\frac{x^2}{4a}$. V is the mid-point of the chord PQ and T is the point of intersection of the tangents at P and Q. Shew that TV is parallel to OY. [Take co-ordinates of P $\left(p, \frac{p^2}{4a}\right)$ and of Q $\left(q, \frac{q^2}{4a}\right)$.]

3. ABCD is a rectangle. On AB, on the side remote from CD, an equilateral triangle ABE is drawn. If AB$=x''$ and BC$=y''$ write down (i) the perimeter, (ii) the area of the figure EADCB.

If the perimeter is 33 inches, what is the maximum area of the figure?

4. A man 6 ft. high walks at a uniform rate of 5 ft./sec. away from a lamp 10 ft. high. Find (i) the rate at which the length of his shadow grows, (ii) the rate at which the end of his shadow moves.

5. The formula for the velocity of sound is

$$v=\sqrt{\frac{\text{Volume Elasticity}}{\text{Density}}}.$$

[If the volume is in c.c., the pressure in dynes per sq. cm. and the density in gms. per c.c., this gives the velocity in cms./sec.]

Find the velocity of sound in air, at 0° C., the height of the mercury barometer being 76 cms., given that density of air under these conditions is ·001293 gms. and that the sp. gr. of mercury is 13·6. The pressure and volume of the air are supposed to be connected by the formula

$$pV^{1\cdot41} = \text{const.}$$

J.

1. A body moves in a straight line in such a way that its acceleration t seconds after it has passed a fixed point O on the line is $(3t+2t^3)$ ft./sec.^{2} Find its distance from O after 5 seconds, given that its speed when it passes O is 10 ft./sec.

2. A uniform sphere diameter $2\frac{1}{16}$ inches is turned down uniformly to diameter $2\frac{1}{32}$ inches. Find approximately the percentage by which its weight is reduced. [Use no tables.]

3. Shew that the expression $1+12x-x^3$ increases in value as x increases from -2 to $+2$ and after that diminishes as x increases.

4. In a certain curve the subtangent at any point is always equal to 3 times the abscissa of the point. Express this as a relation between the (x, y) and $\frac{dy}{dx}$ of the point. [If the tangent at P meet OX in T and PN is the ordinate of P, TN is the subtangent.]

Shew that $y=2x^{\frac{1}{3}}$ fulfils the above condition.

5. A steamer has to go 20 miles up a river which is flowing at 3 miles an hour. The resistance of the water is proportional to the square of the relative velocity and the coal consumed in a given time is proportional to the product of this resistance into the distance the steamer has moved relative to water. If the cost of the coal is to be made a minimum, find the speeds of the steamer relative to shore and relative to water and the time taken on the journey.

CHAPTER VII

INTEGRAL CALCULUS. AREAS UNDER PLANE CURVES. MEAN ORDINATE

134. Integration. Suppose we want the area of the figure bounded by HK part of the curve $y = f(x)$, the x-axis, and the ordinates AH, BK corresponding to $x = a$, $x = b$.

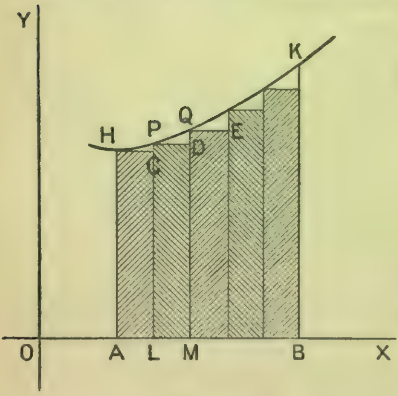


Fig. 63.

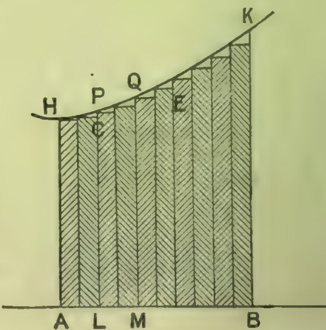


Fig. 64.

Divide AB into any number of equal parts and complete the rectangles as shewn in Fig. 63.

We get an approximation to the area we require by taking it as equal to the sum of these rectangles. This is too small by the sum of the blank portions HCP, PDQ, etc.

If AB were divided into twice as many parts we should get a better approximation to the area, the sum of the 10 rectangles in Fig. 64 being nearer to the area sought than the sum of the 5 rectangles in Fig. 63 [the sum of the two blank portions between H and P < the portion HCP in the previous figure].

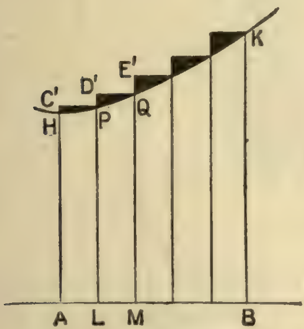


Fig. 65.

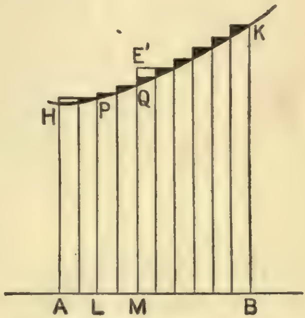


Fig. 66.

By dividing AB into more and more parts we obtain a series of closer and closer approximations to the area sought.

135. If we complete rectangles as in Fig. 65, we get an approximation to the area required by taking the sum of these external rectangles. This is too large by the sum of the black portions HC′P etc.

If AB were divided into twice as many parts we should get a better approximation, the sum of the 10 rectangles in Fig. 66 being nearer to the area sought than the sum of the 5 rectangles in Fig. 65.

[The sum of the two small pieces between H and P in Fig. 66 < the small piece HC′P in Fig. 65.]

By dividing AB into more and more parts we obtain a series of closer and closer approximations to the area sought.

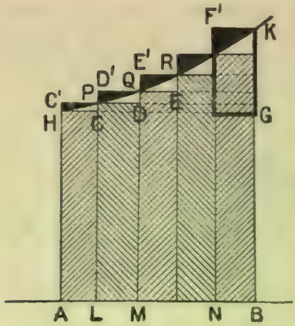


Fig. 67.

136. Combining Figs. 63 and 65 into one (Fig. 67) we see that the difference between the sum of the outside and the sum of the inside rectangles is the sum of the rectangles CC′, DD′, EE′, FF′, = rectangle GF′.

If we increase the number of parts into which AB is divided the area of GF′ diminishes, for its height remains constant and its breadth diminishes, and by making the number of divisions large enough, we can make the area of GF′ as small as we please.

i.e. by taking a sufficiently large number of divisions, we can make the difference between the sum of the outside rectangles and the sum of the inside rectangles as small as we please; and since the area sought lies between these two sums, it is the limit towards which the sum of outside or inside rectangles tends as the number of divisions is indefinitely increased, and the breadth of each division consequently indefinitely diminished.

The determination of this limit is called **Integration.**

137. Note. If the shape of the curve is like that shewn in Figs. 68 and 69, the sum of the first set of rectangles is too large and the sum of the second set too small, otherwise the argument is the same and the area sought is the limit towards which the sum of either set tends as the number of divisions is indefinitely increased.

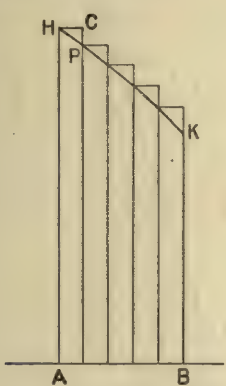


Fig. 68.

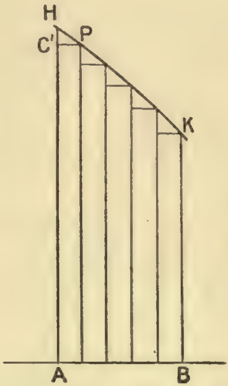


Fig. 69.

If the shape is as in Figs. 70 and 71, we cannot say with certainty of the sum of either set that it is too great or too small,

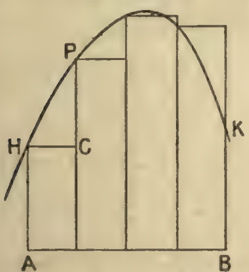


Fig. 70.

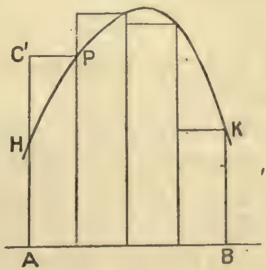


Fig. 71.

for in each case some of the strips are greater and some less than the corresponding portions of the required area.

By dividing the area as in Fig. 72, we can apply the theorem to each part separately.

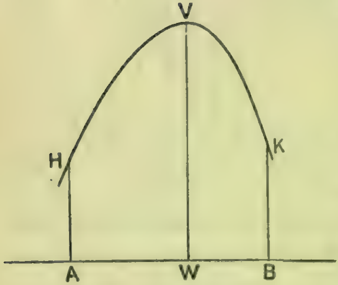


Fig. 72.

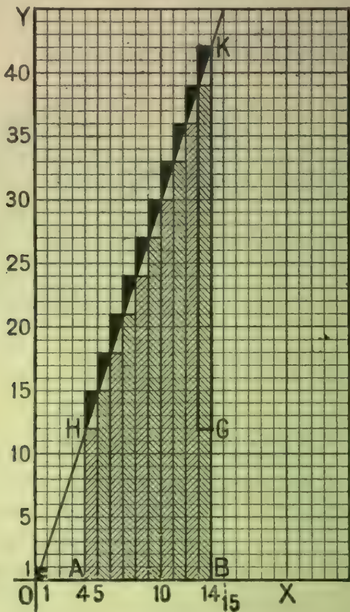


Fig. 73.

138. To make the principle clear we will take an easy special case.

HK is a portion of the line $y = 3x$. (Fig. 73.)

AH, BK are the ordinates corresponding to $x = 4$ and $x = 14$.

Divide AB into 10 equal parts and complete the rectangles as shewn.

The different ordinates are the values of y corresponding to

$$x = 4, 5, \dots 14.$$

Now $y=3x$. Hence the ordinates are 12, 15, 18 ... 42.

The width of each rectangle = 1.

∴ sum of inside rectangles

$$=12+15+18+\ldots+39 \text{ (10 terms)}$$

$$=\frac{10\times 51}{2}=255.$$

Sum of outside rectangles

$$=15+18+\ldots+42 \text{ (10 terms)}$$

$$=\frac{10\times 57}{2}=285.$$

The difference between these is 30, the area of the rectangle GK. [Height $30=\text{BK}-\text{AH}$, width 1.]

The area ABKH lies **between 255 and 285.**

Now suppose AB divided into 100 equal parts.

The different ordinates will be the values of y corresponding to $x=4, 4{\cdot}1, 4{\cdot}2, \ldots 13{\cdot}9, 14$.

∴ the ordinates are

$$12,\ 12{\cdot}3,\ 12{\cdot}6, \ldots 41{\cdot}7,\ 42.$$

The width of each rectangle is ·1.

∴ sum of inside rectangles

$$=\cdot 1\,[12+12{\cdot}3+12{\cdot}6+\ldots+41{\cdot}7] \text{ 100 terms}$$

$$=\cdot 1\times\frac{100\times 53{\cdot}7}{2}=\mathbf{268{\cdot}5},$$

and sum of outside rectangles

$$=\cdot 1\,[12{\cdot}3+12{\cdot}6+\ldots+42]$$

$$=\cdot 1\times\frac{100\times 54{\cdot}3}{2}=\mathbf{271{\cdot}5}.$$

The difference between these is 3, the area of a rectangle, height 30 and width ·1.

The area ABKH lies **between 268·5 and 271·5.**

Similarly shew that if AB be divided into 1000 equal parts the area ABKH lies **between 269·85 and 270·15.**

We have two sets of numbers:

$$255,\ 268{\cdot}5,\ 269{\cdot}85, \ldots$$

$$285,\ 271{\cdot}5,\ 270{\cdot}15, \ldots$$

apparently both continually approaching a limit 270.

To make quite sure of this, suppose AB divided into n equal parts each $= h$, so that $nh = 10$.

The ordinates are the values of y corresponding to

$$x = 4,\ 4 + h,\ 4 + 2h,\ \ldots,\ 14 - h,\ 14.$$

Hence the ordinates are

$$12,\ 12 + 3h,\ 12 + 6h,\ \ldots, 42 - 3h,\ 42.$$

The width of each rectangle is h.

$\therefore$ sum of inside rectangles

$$= h\left[12 + (12 + 3h) + (12 + 6h) + \ldots + (42 - 3h)\right] (n \text{ terms})$$

$$= h \,.\, \frac{n}{2}(54 - 3h) = \frac{nh}{2}(54 - 3h) = \frac{10}{2}(54 - 3h) = \mathbf{270 - 15h} \ldots(1),$$

and sum of outside rectangles

$$= h\left[(12 + 3h) + (12 + 6h) + \ldots + 42\right]$$

$$= h \,.\, \frac{n}{2}(54 + 3h) = \mathbf{270 + 15h} \ \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The difference is $30h$ the area of a rectangle height 30, width h.

The area ABKH lies **between 270 − 15h and 270 + 15h.**

Now, as the number of strips is indefinitely increased, i.e. as $h \to 0$, each of the areas (1) and (2) can be made as near as we please to 270.

$\therefore$ the area ABKH which is the limit of either sum is **270.**

Since the limit of the sum of inside rectangles when $h \to 0$ is the same as the limit of the sum of outside rectangles, we need only calculate one of the sums.

We may say

either sum of inside rectangles $= 270 - 15h$,

$\therefore$ area reqd. = Limit of this when $h \to 0, = 270$;

or sum of outside rectangles $= 270 + 15h$,

$\therefore$ area reqd. = Limit of this when $h \to 0, = 270$.

139. Note on units. In Fig. 73 the x-scale and the y-scale are the same, and the unit of area is the small square which is shaded, whose side is the unit of length.

The area ABKH is 270 times this unit.

$\therefore$ if the unit be ·1″ each way, the unit of area is ·01 sq. in. and area ABKH $= 270 \times \cdot 01 = 2\cdot 7$ sq. ins.

If however the scales are different as in Fig. 74 the unit of area is no longer a square but a rectangle whose sides are equal respectively to the x-unit and the y-unit. The area ABKH is 270 times this rectangle.

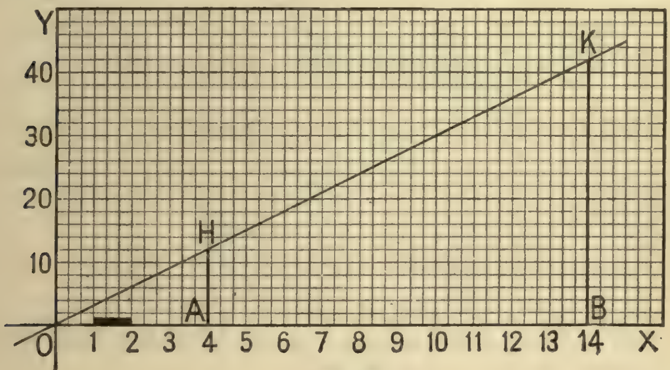


Fig. 74.

If x-unit = ·3″ and y-unit = ·05″,

unit of area = ·015 sq. ins.,

and area ABKH $= 270 \times \cdot 015 = 4\cdot 05$ sq. ins.

EXERCISES. XXXV.

1. Find by the method of § 138 the area bounded by

$$y=5x+7,\ y=0,\ x=0,\ x=5.$$

If the unit is $\frac{1}{10}''$ each way what is the area in sq. ins.?

What is it if the x-unit is $\frac{1}{2}''$ and the y-unit $\frac{1}{10}''$?

2. Find the area bounded by

$$y=3x+2,\ x=0,\ y=3,\ y=7.$$

[Divide into strips parallel to Ox.]

What is the area in sq. ins. if the x-unit is $3''$ and the y-unit $\frac{1}{2}''$?

3. P is the point (10,100) on the curve $y=x^2$. PN is the ordinate of P and O is the origin.

Find the area OPN bounded by ON, NP and the curve.

(i) Divide ON into 10 equal parts,

(ii) ,, ,, ,, 100 ,, ,,

(iii) ,, ,, ,, n ,, ,, each h,

and in every case find the sum of the inside and outside rectangles, using the formula

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

(iv) Find the limit of each of the last results as $h \to 0$.

4. Find the area OPN bounded by $y=x^3$, the x-axis and the ordinate $x=3$.

Divide ON into n equal parts and use the formula

$$1^3+2^3+3^3+\ldots+n^3=\left\{\frac{n(n+1)}{2}\right\}^2.$$

140. We shall now solve the problem of § 138 by adopting an entirely different point of view.

P is any point (x, y) on the line $y=3x$. (Fig. 75.)

An ordinate is supposed to advance from the position AH corresponding to $x=4$, to the position BK corresponding to $x=14$, its length continually changing in such a way that its upper end

is always in the line HK. When the ordinate reaches MP, it has swept out a certain area which is obviously a function of x, i.e. it is determined when x is known and changes as x changes. Call this area A.

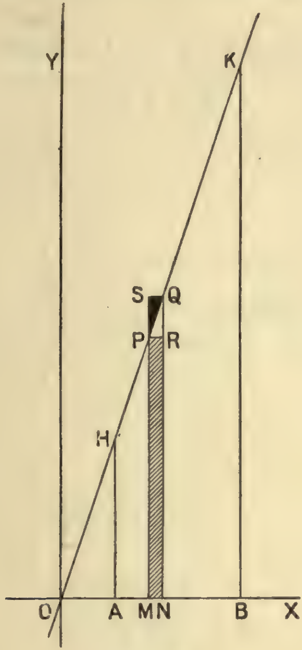


Fig. 75.

We will consider the change in the area due to a short advance of the ordinate.

Let Q be a point $(x+\Delta x, \ y+\Delta y)$ on the line and complete the rectangles as in the figure.

Then the area PMNQ is the increment in the area A, due to an increment Δx in x, and may therefore be denoted by ΔA.

Now PMNQ lies between PMNR and SMNQ.

i.e. ΔA lies between $y \,.\, \Delta x$ and $(y + \Delta y) \,.\, \Delta x$.

$$\therefore \quad \frac{\Delta A}{\Delta x} \text{ lies between } y \text{ and } (y + \Delta y).$$

As Δx, and with it $\Delta y \to 0$, $(y + \Delta y) \to y$.

i.e. $$\frac{dA}{dx} = y = 3x.$$

$\therefore$ A is a function of x which when differentiated gives $3x$.

$$\therefore \quad A = \frac{3}{2}x^2 + c$$

where c is some constant to be determined.

Now since A stands for the area traced out by an ordinate starting from AH, $A = 0$ when the ordinate coincides with AH, i.e. when $x = 4$.

$$\therefore \quad 0 = \frac{3}{2} \times 4^2 + c.$$

$$\therefore \quad c = -24.$$

$$\therefore \quad A = \frac{3}{2}x^2 - 24.$$

We want the value of A corresponding to $x = 14$.

$$\therefore \quad A = \frac{3}{2} \times 14^2 - 24$$
$$= 294 - 24$$
$$= 270.$$

Notice 294 is the value of $\frac{3}{2}x^2$ when $x = 14$

and 24 is the value of $\frac{3}{2}x^2$ when $x = 4$,

and the area sought is the difference of these.

EXERCISES. XXXVI.

To be solved by the method of § 140.

1. Find the area bounded by $y=3x$, $y=0$, $x=3$, $x=9$.
2. Find the area bounded by $y=3x$, $y=0$, $x=0$, $x=9$.
3. Find the area bounded by $y=3x$, $y=0$, $x=0$, $x=3$.
4. Find the area bounded by $y=\frac{1}{2}x+2$, $y=0$, $x=5$, $x=10$.
5. Find the area bounded by $y=5x+7$, $y=0$, $x=0$, $x=5$.
6. Find the area bounded by $y=3x+2$, $x=0$, $y=3$, $y=7$.
7. Find the area bounded by $y=4x-3$, $x=0$, $y=0$, $y=11$.

141. *Example.* Find the area OBK bounded by $y=0$, $x=10$ and part of $y=x^2$.

As before let MP be the ordinate of the point P (x, y) and PMNQ a strip of width Δx. [Fig. 76.]

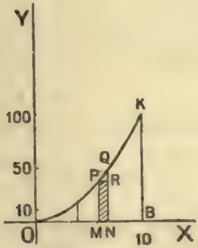


Fig. 76.

Let A be the area OMP, then ΔA lies between $x^2 \Delta x$ and $(x+\Delta x)^2 \Delta x$.

$$\therefore \quad \frac{dA}{dx}=x^2.$$

$$\therefore \quad A=\frac{x^3}{3}+c.$$

This time our moving ordinate which is supposed to trace out the area starts from O where $x = 0$.

$\therefore$ when $x = 0$, $A = 0$.

$$\therefore \; c = 0.$$

$$\therefore \; A = \frac{x^3}{3}.$$

$$\therefore \text{ area OBK} = \frac{10^3}{3} = 333\tfrac{1}{3}.$$

EXERCISES. XXXVII.

1. Find the area bounded by $y = x^2$, $y = 0$, $x = 3$, $x = 14$.
2. Find the area in the first quadrant bounded by
$$y = x^2, \; x = 0, \; y = 1, \; y = 4.$$
3. Find the area bounded by $y = x^3$, $y = 0$, $x = 2$, $x = 5$.
4. Find the area bounded by $y = 2x^2 + 3x + 1$, $y = 0$, $x = 3$, $x = 7$.
5. Find the area bounded by $y = x^3 + x$, $y = 0$, $x = 3$, $x = 11$.
6. Find the area bounded by $y = x^3$, $x = 0$, $y = 8$, $y = 27$.
7. Find the area bounded by $y = x^3$, $y = 0$, $x = a$, $x = b$.

142. Notice that in leading up to the equation

$$\frac{dA}{dx} = 3x \text{ in § 140}$$

we could have supposed the moving ordinate to start from any position. The fact that it was supposed to start from the position AH was used in the determination of the constant c in

$$A = \frac{3}{2}x^2 + c.$$

The equation
$$A = \frac{3}{2}x^2 - 24$$

is a formula for the area traced out by an ordinate which starts from AH and moves into any other position.

Suppose now that the moving ordinate started from A′H′ ($x=2$). [Fig. 77.]

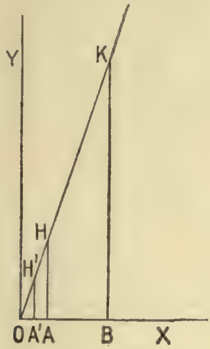


Fig. 77.

We should get in the same way

$$A=\frac{3}{2}x^2-6$$

as a formula for the area traced out by an ordinate which starts from A′H′ and moves into any other position.

Thus
$$\text{area } A'H'HA=\frac{3}{2}\cdot 4^2-6,$$

$$\text{area } A'H'KB=\frac{3}{2}\cdot 14^2-6.$$

$$\therefore \text{ area } AHKB=\left(\frac{3}{2}\cdot 14^2-6\right)-\left(\frac{3}{2}\cdot 4^2-6\right)$$

$$=\frac{3}{2}\cdot 14^2-\frac{3}{2}\cdot 4^2,$$

as before, the constant -6 disappearing in the subtraction.

Therefore if we are finding the area between two given ordinates $x=4$, $x=14$ we can deduce at once from

$$\frac{dA}{dx}=3x$$

that area between $x=4$ and $x=14$

$$=\frac{3}{2}.\ 14^2-\frac{3}{2}.\ 4^2,$$

or as it may be written $\left[\frac{3}{2}x^2\right]_4^{14}$;

and the same result would be obtained if we wrote for

$$\frac{3}{2}x^2,\ \frac{3}{2}x^2+c$$

where c is any constant whatever.

In this particular case, $A=\frac{3}{2}x^2$ is a formula for the area traced out by an ordinate which starts from the origin, so that when we say

$$\text{area AHKB}=\left[\frac{3}{2}x^2\right]_4^{14}=\left[\frac{3}{2}x^2-6\right]_4^{14}$$

we are merely saying that

$$\text{AHKB}=\text{OKB}-\text{OHA}=\text{A}'\text{H}'\text{KB}-\text{A}'\text{H}'\text{HA}.$$

143. In Ex. XXXVII (4), the result is $309\frac{1}{6}-34\frac{1}{2}$, $309\frac{1}{6}$ and $34\frac{1}{2}$ being the values of $\frac{2x^3}{3}+\frac{3x^2}{2}+x$ when $x=7$ and $x=3$ respectively.

$\frac{2x^3}{3}+\frac{3x^2}{2}+x$ gives the area traced out by an ordinate starting from OL.

$309\frac{1}{6}$ is area OLKB; $34\frac{1}{2}$ is area OLHA. [Fig. 78.]

We might have said

$$\text{area}=\left[\frac{2x^3}{3}+\frac{3x^2}{2}+x-13\frac{1}{3}\right]_3^7$$

$$=295\tfrac{5}{6}-21\tfrac{1}{6},$$

$295\frac{5}{6}$ being area A′H′KB and $21\frac{1}{6}$ the area A′H′HA, where A′H′ is the ordinate $x=2$, the constant $13\frac{1}{3}$ being chosen so that

$$\frac{2x^3}{3}+\frac{3x^2}{2}+x-13\tfrac{1}{3}=0 \text{ when } x=2.$$

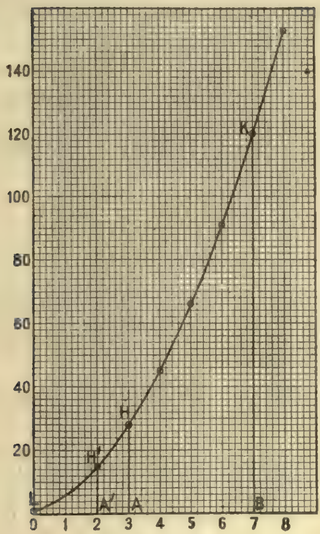


Fig. 78.

144. In Ex. XXXVII (7) the result is $\frac{b^4}{4}-\frac{a^4}{4}$, $\frac{b^4}{4}$ and $\frac{a^4}{4}$ being the values of $\frac{x^4}{4}$ when $x=b$ and $x=a$ respectively. We might write the result $A=\left[\frac{x^4}{4}\right]_a^b$, meaning

$$\left[\text{Value of } \frac{x^4}{4} \text{ when } x=b\right]-\left[\text{Value of } \frac{x^4}{4} \text{ when } x=a\right].$$

and the work might be stated shortly thus:

$$\frac{dA}{dx}=x^3,$$

$$\therefore\ A=\left[\frac{x^4}{4}\right]_a^b=\frac{b^4}{4}-\frac{a^4}{4},$$

$\frac{x^4}{4}$ being a function [regardless of arbitrary constant] which when differentiated gives x^3.

145. Notice in Ex. (4), instead of finding the values of

$$\frac{2x^3}{3}+\frac{3x^2}{2}+x$$

when $x=7$ and when $x=3$ we might more conveniently say

$$\left[\frac{2x^3}{3}+\frac{3x^2}{2}+x\right]_3^7=\frac{2}{3}(7^3-3^3)+\frac{3}{2}(7^2-3^2)+(7-3)$$

$$=\frac{2}{3}.\,316+\frac{3}{2}.\,40+4$$

$$=274\tfrac{2}{3}.$$

146. Generally. If we want the area bounded by $y=f(x)$ [any function of x], $y=0$, $x=a$, $x=b$, we have, making the same construction as in § 140:

Area of strip PMNQ lies between $y\Delta x$ and $(y+\Delta y)\Delta x$. [Fig. 79.]

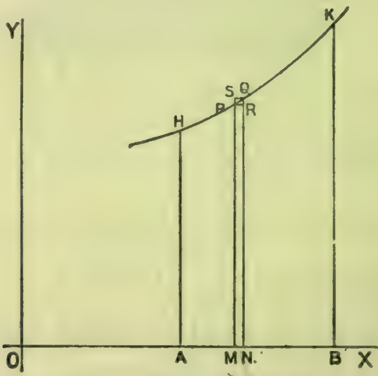


Fig. 79.

i.e. ΔA is between $y\Delta x$ and $(y+\Delta y)\Delta x$.

$$\therefore\quad \frac{\Delta A}{\Delta x} \text{ is between } y \text{ and } y+\Delta y;$$

but as Δx, and with it Δy, $\to 0$, $y+\Delta y \to y$,

$$\therefore\quad \frac{dA}{dx}=y=f(x).$$

$\therefore$ A is some function of x which when differentiated gives $f(x)$. Suppose $\phi(x)$ to be such a function,

$$\therefore \quad A = \phi(x) + c.$$

Now $A = 0$ when $x = a$, $\quad \therefore \; c = -\phi(a)$, $\quad \therefore \; A = \phi(x) - \phi(a)$,

$$\therefore \text{ area between AH and BK} = \phi(\mathrm{b}) - \phi(\mathrm{a}) = \Big[\phi(\mathrm{x})\Big]_{\mathrm{a}}^{\mathrm{b}}.$$

147. Notice that PMNR is the area which would be traced out by the moving ordinate between M and N if it kept throughout the length which it has at M.

SMNQ is the area which would be traced out if it kept throughout the length which it has at N.

Notice also that so far as finding $\dfrac{dA}{dx}$ is concerned, it is sufficient to say

$$\Delta A = f(x) \,.\, \Delta x \text{ approximately,}$$

$$\therefore \frac{\Delta A}{\Delta x} = f(x) \text{ approximately,}$$

$$\therefore \frac{dA}{dx} = f(x),$$

it being understood that the statement

$$\Delta A = f(x) \,.\, \Delta x \text{ approximately}$$

means that ΔA is between $f(x) \,.\, \Delta x$ and $f(x + \Delta x) \,.\, \Delta x$.

EXERCISES. XXXVIII.

Find formulae for the area traced out by an ordinate of $y = 5x^3 + 2$ which starts from the position given by (i) $x = 0$, (ii) $x = 3$, (iii) $x = -2$ and moves into any other position. Deduce from each of these formulae the area traced out by an ordinate which moves from $x = 5$ to $x = 10$.

148. Returning to the investigation in § 138, p. 166, each term in the series (1) such as $(12 + 6h)\,h$ is the area of a rectangular strip.

$(12 + 6h)$ is the value of y or $3x$ when $x = 4 + 2h$.

The series is

$$12h + (12 + 3h)\,h + (12 + 6h)\,h + \ldots + (42 - 3h)\,h;$$

$$[12,\ 12 + 3h,\ 12 + 6h, \ldots (42 - 3h),$$

being the values of y or $3x$ when $x = 4, 4 + h, 4 + 2h, \ldots (14 - h)$], or more shortly

$$\sum_{x=4}^{x=14-h} 3x\,.\,h,$$

meaning "Take values of x, starting at 4 and increasing by h at a time up to $(14 - h)$; for each value of x, find the corresponding value of $3x$; multiply each of these values by h, and add all the results."

The actual area sought is the limit of this sum when $h \to 0$.

$\sum\limits_{x=4}^{x=14} 3x\,.\,h$ would mean the same as $\sum\limits_{x=4}^{x=14-h} 3x\,.\,h$ with the addition of an extra strip **42h** (42 being the value of $3x$ when $x = 14$) and this can be made as small as we please by taking h small enough.

i.e. the limits when $h \to 0$ of $\sum\limits_{x=4}^{x=14} 3x\,.\,h$ and $\sum\limits_{x=4}^{x=14-h} 3x\,.\,h$ are the same.

$\therefore$ we may say that the area ABKH is $\underset{h \to 0}{\mathrm{Lt}} \sum\limits_{x=4}^{x=14} 3xh.$

149. Generally if the curve is $y = f(x)$, adopting the usual notation as in § 146, the area of the rectangular strip PMNR is $y\Delta x$, and the sum of the areas of all such strips between AH and BK may be denoted by $\sum\limits_{x=a}^{x=b} y\Delta x$, meaning: take values of x starting at a and increasing Δx at a time up to b; for each value of x get the corresponding value of y from $y = f(x)$; multiply each value of y by Δx and add all the results.

The area of the figure bounded by HA, AB, BK and the curve is the limit to which this sum tends as $\Delta x \to 0$ and this is

denoted by $\int_a^b ydx$ $\left[\int\right.$ being simply a lengthened form of the letter S$\left.\right]$.

150. Thus $\int_a^b ydx$ is defined to be $\underset{\Delta x \to 0}{\text{Lt}} \sum_{x=a}^{x=b} y\Delta x$, and we have seen that in order to get this limit, we must find a function of x which when differentiated gives y, and subtract the value of this function when $x = a$ from its value when $x = b$.

i.e. $$\int_a^b f(x).dx = \phi(b) - \phi(a) \text{ or } \Big[\phi(x)\Big]_a^b,$$

where $$\frac{d.\phi(x)}{dx} = f(x).$$

e.g. $$\int_4^{14} (3x)\,dx = \left[\frac{3x^2}{2}\right]_4^{14} = \frac{3}{2}(14^2 - 4^2),$$

that is to say instead of finding the area by a process of direct summation, we find the rate at which the area increases with respect to x and from this deduce the area as a function of x.

In other words instead of finding A from

$$A = \int_a^b f(x)\,dx = \underset{\Delta x \to 0}{\text{Lt}} \sum_{x=a}^{x=b} f(x)\,\Delta x,$$

a summation which in most cases it would be impossible or at any rate inconvenient to effect, we find A from

$$\frac{dA}{dx} = f(x)$$

and we say that

$$\int_a^b f(x).dx = \Big[\phi(x)\Big]_a^b$$

where $\phi(x)$ is a function of x such that

$$\frac{d\phi(x)}{dx} = f.(x).$$

151. Notice that just as dy and dx in the expression $\frac{dy}{dx}$ have no separate meanings, but the form $\frac{dy}{dx}$ is preserved to

remind us of $\frac{\Delta y}{\Delta x}$ of which $\frac{dy}{dx}$ is the limit, so dx in the expression $\int_4^{14}(3x)\,dx$ has no separate meaning, but the form $\int_4^{14}(3x)\,dx$ is preserved to remind us of $\sum_{x=4}^{x=14}(3x)\,\Delta x$ of which it is the limit. Δx represents a definite length and $\sum_{x=4}^{x=14}(3x)\,\Delta x$ represents the sum of a **finite** number of rectangles.

$\int_4^{14}(3x)\,dx$ stands for the limit of this sum as $\Delta x \to 0$.

152. An expression like $\int_4^{14}(3x)\,dx$ is called a **definite integral**, 4 and 14 being called the limits of the integral.

It is read "Integral of $(3x)\,dx$ between 4 and 14."

We have seen that in order to find the value of this we must first discover a function of x which when differentiated gives $(3x)$.

This function is written $\int(3x)\,dx$ and is read "Integral $(3x)\,dx$."

Thus
$$\int(3x)\,dx = \frac{3}{2}x^2 + c$$
$$\int_4^{14}(3x)\,dx = \left[\frac{3}{2}x^2 + c\right]_4^{14}$$
$$= \frac{3}{2}(14^2 - 4^2).$$

An expression like $\int(3x)\,dx$ is called an **indefinite integral.**

$$y = \int(3x)\,dx,$$
$$\frac{dy}{dx} = 3x$$

are merely different forms of the same statement. When the

value of an indefinite integral has been found, it should always be checked by differentiation.

e.g. $$\int\left(3x^3 - x^{\frac{1}{2}} + \frac{2}{x^2}\right) dx = \frac{3x^4}{4} - \frac{2}{3} \cdot x^{\frac{3}{2}} - \frac{2}{x} + c.$$

Check :

$$\frac{d}{dx}\left(\frac{3x^4}{4} - \frac{2}{3}\, x^{\frac{3}{2}} - \frac{2}{x} + c\right) = \frac{3}{4} \cdot 4x^3 - \frac{2}{3} \cdot \frac{3}{2} \cdot x^{\frac{1}{2}} - 2\left(-\frac{1}{x^2}\right) + 0$$

$$= 3x^3 - x^{\frac{1}{2}} + \frac{2}{x^2}.$$

The first step in the evaluation of a definite integral is the determination of the indefinite integral*.

EXERCISES. XXXIX.

Write down the values of the following indefinite integrals and in each case check by differentiation:

1. $\int 2x^2\, dx.$
2. $\int\left(3\sqrt{x} + \frac{1}{x^3}\right) dx.$
3. $\int (ax^2 + bx + c)\, dx.$
4. $\int \frac{1}{3\sqrt[3]{x}} \cdot dx.$
5. $\int (5t + 6t^2 - t^{\frac{2}{3}})\, dt.$
6. $\int \frac{1}{v^{1.4}}\, dv.$
7. $\int 7dx.$
8. $\int\left(3x^3 - 5x^2 + 2x - 3 + \frac{1}{3x^2}\right) dx.$
9. $\int\left(x + \frac{1}{x^2}\right) dx.$

* Notice that the indefinite integral always contains an arbitrary constant, but in the subtraction which forms part of the process of evaluating the definite integral, this constant disappears and may therefore be omitted, so that, instead of saying

$$\int_4^{14} (3x)\, dx = \left[\frac{3}{2}x^2 + c\right]_4^{14},$$

we may say

$$\int_4^{14} (3x)\, dx = \left[\frac{3}{2}x^2\right]_4^{14}.$$

Hence write down the values of the following definite integrals :

10. $\displaystyle\int_2^7 (2x^2)\,dx.$

11. $\displaystyle\int_1^2 \left(3\sqrt{x}+\frac{1}{x^3}\right) dx.$

12. $\displaystyle\int_{-1}^1 (ax^2+bx+c)\,dx.$

13. $\displaystyle\int_1^8 \frac{1}{3\sqrt[3]{x}}\,dx.$

14. $\displaystyle\int_0^8 (5t+6t^2-t^{\frac{2}{3}})\,dt.$

15. $\displaystyle\int_{30}^{40} \frac{1}{v^{1\cdot 4}}\,dv.$

16. $\displaystyle\int_{-3}^5 7dx.$

17. $\displaystyle\int_3^6 \left(3x^3-5x^2+2x-3+\frac{1}{3x^2}\right) dx.$

18. $\displaystyle\int_1^2 \left(x+\frac{1}{x^2}\right) dx.$

Find the values of the following:

19. $\displaystyle\sum_{x=2}^{x=7} 2x^2 . \Delta x$ (i) when $\Delta x = 1$, (ii) when $\Delta x = \cdot 05$.

20. $\displaystyle\sum_{x=-3}^{x=5} 7 . \Delta x$ (i) when $\Delta x = \cdot 1$, (ii) when $\Delta x = \cdot 01$.

21. $\displaystyle\sum_{x=1}^{x=2} \left(x+\frac{1}{x^2}\right) \Delta x$ when $\Delta x = \cdot 2$.

Compare your results with those of Exs. 10, 16, 18.

153. The process of finding an area may now be written down as follows :—(taking again the example of § 140)

Area of strip $= (3x)\,\Delta x$ app.

$\left\{\therefore\right.$ Sum of strips $= \displaystyle\sum_{x=4}^{x=14} (3x)\,\Delta x$ app. and area required is the limit of this sum when $\Delta x \to 0.\left.\right\}$

$$\therefore\ \text{Area required} = \int_4^{14} (3x)\,dx$$

$$= \left[\frac{3}{2}x^2\right]_4^{14}$$

$$= \frac{3}{2}(14^2 - 4^2) = 270.$$

After a little practice the part in brackets {} may be omitted.

EXERCISES. XL.

1. P is the point (2, 4) on the parabola $y=x^2$. PM and PN are perp. to OX and OY.

Find (i) area OMP, (ii) area ONP.

Verify that their sum is the area of the rectangle OMPN.

2. The same where P is (2, 8) on $y=x^3$.

3. P is the point $\left(h, \frac{h^2}{a}\right)$ on the parabola $y=\frac{x^2}{a}$. PM and PN are perpendicular to OX and OY.

Find separately areas OMP, ONP and shew that they are respectively one-third and two-thirds of the rect. OMPN.

4. P is the point $\left(h, \frac{h^3}{a^2}\right)$ on the curve $y=\frac{x^3}{a^2}$, PM, PN are perp. to OX, OY.

Prove that OMP, ONP are respectively one-quarter and three-quarters of the rect. OMPN.

5. P is the point $\left(h, \frac{h^n}{a^{n-1}}\right)$ on the curve $y=\frac{x^n}{a^{n-1}}$. PM, PN are perpendicular to OX, OY. Prove OMP$=\frac{1}{n+1}$ rect. OMPN.

6. Find the area bounded by

$$y=2x^2-3x+5,\ y=0,\ x=-2,\ x=7.$$

7. Find the area bounded by

$$y=ax^2+bx+c,\ y=0,\ x=-p,\ x=p.$$

8. The co-ordinates of H, K, L are respectively

$$(-a,\ h)\ (a,\ k)\ (0,\ l).\quad \text{[Fig. 80.]}$$

A curve whose equation is $y=p+qx+rx^2$ passes through H, L, K.

Find p, q, r in terms of a, h, k, l.

9. In Qu. 8, if HA, KB are the ordinates of H, K, prove that the area

$$\text{HABK}=\frac{a}{3}\{h+4l+k\}.$$

154. The result of Exs. XL. (9) leads to a very important rule called Simpson's Rule for finding approximately the area of any figure.

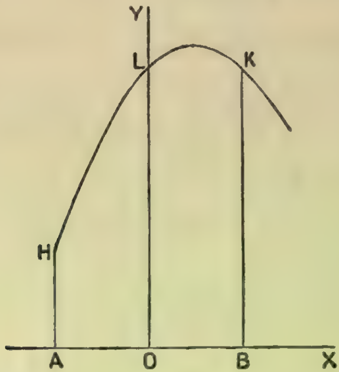


Fig. 80.

Suppose we want the area bounded by the curved line PQRSTUV, the ordinates AP, GV, and AG perp. to AP and GV. [Fig. 81.]

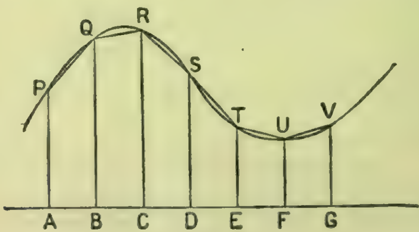


Fig. 81.

We can get a first approximation as follows :—

Divide AG into any number of equal parts and draw ordinates through the points of division as in the figure. Join PQ, QR etc.

Then if AP $= h_1$, BQ $= h_2$, etc. and AB = BC = etc. $= a$ we have

Area of trapezium ABQP $= \frac{a}{2}(h_1 + h_2)$.

Area of trapezium BCRQ $= \frac{a}{2}(h_2 + h_3)$ and so on.

Area of trapezium FGVU $= \frac{a}{2}(h_6 + h_7)$.

$\therefore$ by addition, Area of figure is approximately

$$\frac{a}{2}\{h_1 + h_7 + 2(h_2 + h_3 + h_4 + h_5 + h_6)\}$$

or generally, for any number of equidistant ordinates

Area is approximately $\frac{1}{2}$. Distance between consecutive ordinates $\times$ {sum of extreme ordinates + twice sum of intermediate ordinates}.

This is the **Trapezoidal** rule and its accuracy is obviously increased by taking more and more strips.

Notice that this result is obtained by substituting for the curved boundary, a boundary composed of straight lines through pairs of consecutive points.

In other words the portion of the boundary between two consecutive points is replaced by a line whose equation is of the first degree or of the form $y = p + qx$.

155. Now if we suppose the portion of the boundary passing through 3 consecutive points to be replaced by a curve whose equation is of the second degree of the form $y = p + qx + rx^2$, we shall get a closer approximation.

Considering the portion PQR.

Take B as origin and axes along BC and BQ.

Then co-ordinates of P, Q, R are respectively

$$(-a, h_1)\ (0, h_2)\ (a, h_3)$$

and if $y = p + qx + rx^2$ passes through these three points, we have

$$\left.\begin{aligned} h_1 &= p - qa + ra^2 \\ h_2 &= p \\ h_3 &= p + qa + ra^2 \end{aligned}\right\}$$

whence $$p = h_2, \quad q = \frac{h_3 - h_1}{2a}, \quad r = \frac{h_1 - 2h_2 + h_3}{2a^2}.$$

Now area ACRP bounded by PA, AC, CR and $y = p + qx + rx^2$

$$= \int_{-a}^{a} (p + qx + rx^2)\, dx = \left[px + \frac{qx^2}{2} + \frac{rx^3}{3}\right]_{-a}^{a}$$

$$= 2pa + \frac{2ra^3}{3}.$$

Substituting the values of p and r, this becomes

$$2h_2a + \frac{a}{3}(h_1 - 2h_2 + h_3) = \frac{a}{3}(h_1 + 4h_2 + h_3),$$

and this is taken as being approximately the area of the portion PACR of the given figure.

If there is an even number of strips as in the figure, we shall get in exactly the same way

$$\text{Area RCET} = \frac{a}{3}(h_3 + 4h_4 + h_5) \text{ approx.}$$

and $$\text{Area TEGV} = \frac{a}{3}(h_5 + 4h_6 + h_7) \quad \text{,,}$$

Adding we get

$$\text{Area of figure} = \frac{a}{3}\{h_1 + h_7 + 4(h_2 + h_4 + h_6) + 2(h_3 + h_5)\} \text{ approx.}$$

and similarly for any **even** number of strips.

We thus have Simpson's rule:

Divide the area into an even number of strips of equal width by an odd number of ordinates; the area is approximately

$\frac{1}{3}$. Width of a strip × {sum of extreme ordinates + twice sum of other odd ordinates + 4 times sum of even ordinates}.

EXERCISES. XLI.

1. Find by integration the area bounded by $y=0$, $x=2$, $x=12$ and the curve $y=x^3$.

2. Find the approximate area in (1) by using 3 ordinates and applying (i) the trapezoidal rule, (ii) Simpson's rule.

3. Do the same as in (2) using 11 ordinates.

4. Do the same as in (1), (2) and (3) for the area bounded by $y=0$, $x=2$, $x=7$ and $y=x^4$.

156. Note on sign of area. If we advance from left to right, i.e. if Δx is positive, $y\Delta x$ representing the area of our typical strip is positive if y is positive, i.e. if the strip lies above the x-axis and negative if the strip lies below the x-axis [Fig. 82.]

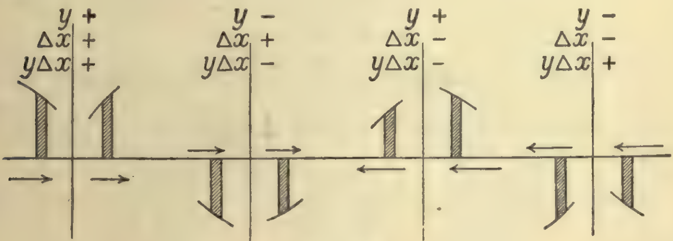


Fig. 82.

If we advance from right to left, i.e. if Δx is negative, $y\Delta x$ is positive if the strip lies below and negative if the strip lies above the x-axis.

157. Thus if the curve be $y=\frac{x^3}{a^2}$ and P, Q be respectively the points $(-a,-a)$, (a,a) [Fig. 83], $\int_{-a}^{0}\frac{x^3}{a^2}\,dx$ will give the area PMO and will be negative since we are advancing from left to right

and the area is below the axis, i.e. during the whole motion of the ordinate from MP to O, Δx is + and y is −.

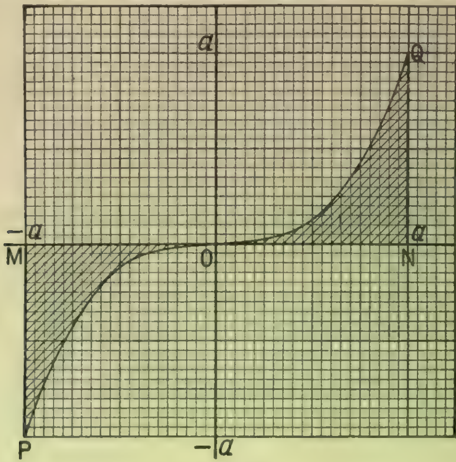


Fig. 83.

$\int_0^{-a} \frac{x^3}{a^2}\,dx$ will give the area OMP and will be positive since we are advancing from right to left and the area is below the axis, i.e. Δx is − and y is −.

Similarly $\int_0^{a} \frac{x^3}{a^2}\,dx$ will be positive and $\int_a^{0} \frac{x^3}{a^2}\,dx$ will be negative.

If we take $\int_{-a}^{a} \frac{x^3}{a^2}\,dx$ we shall get zero, which simply means that the moving ordinate in passing from MP to NQ sweeps out two numerically equal areas of opposite signs.

If we want the actual area shaded, we must find one of the portions PMO, ONQ separately.

$$\text{ONQ} = \int_0^{a} \frac{x^3}{a^2}\,dx = \left[\frac{x^4}{4a^2}\right]_0^a = \frac{a^2}{4}.$$

$$\therefore \textbf{ Shaded area} = \frac{a^2}{2}.$$

158. As another example, consider the curve

$$y = 4 + 3x - x^2,$$

between $x = -2$ and $x = +5$. [Fig. 84.]

The area bounded by the curve, the x-axis, and the extreme ordinates is seen from the figure to consist of 3 parts, two below the x-axis and one above.

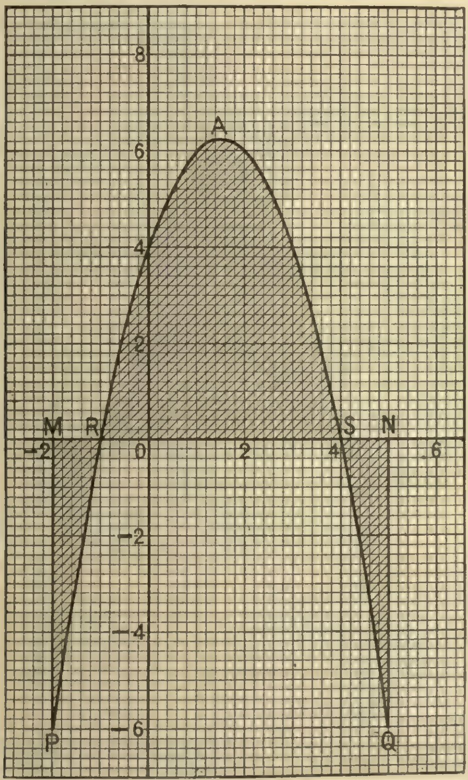


Fig. 84.

Now $\int_{-2}^{5} (4+3x-x^2)\,dx = \left[4x+\frac{3x^2}{2}-\frac{x^3}{3}\right]_{-2}^{5}$

$$= 4\,(7) + \frac{3}{2}\,.\,21 - \frac{1}{3}\,.\,133$$

$$= 15\tfrac{1}{6}.$$

The co-ordinates of the points R and S are $(-1, 0)$ and $(4, 0)$.

$$\int_{-2}^{-1}(4+3x-x^2)\,dx = \left[4x+\frac{3}{2}x^2-\frac{1}{3}x^3\right]_{-2}^{-1}$$

$$= 4\,.\,1 + \frac{3}{2}(-3) - \frac{1}{3}(7)$$

$$= -2\tfrac{5}{6}.$$

i.e. actual area of portion PMR $= 2\frac{5}{6}$.

$$\int_{-1}^{4}(4+3x-x^2)\,dx = 4\,.\,5 + \frac{3}{2}\,.\,15 - \frac{1}{3}\,.\,65$$

$$= 20\tfrac{5}{6}.$$

This is area RAS.

$$\int_{4}^{5}(4+3x-x^2)\,dx = 4\,.\,1 + \frac{3}{2}\,.\,9 - \frac{1}{3}\,.\,61$$

$$= -2\tfrac{5}{6}.$$

i.e. actual area of portion SNQ $= 2\frac{5}{6}$.

And $15\frac{1}{6} = 20\frac{5}{6} - 2\frac{5}{6} - 2\frac{5}{6}$.

i.e. the definite integral gives the algebraic sum of the 3 areas PMR, RAS, SNQ, or the excess of the actual area PMR over the sum of the actual areas of RAS and SNQ.

EXERCISES. XLII.

1. Shew that $\int_0^2 (1-x)\,dx = 0$.

Draw a figure to explain the result.

2. Shew that $\int_0^2 (x^3-3x^2+2x)\,dx = 0$.

Draw a figure to explain the result.

If the curve $y=x^3-3x^2+2x$ meet the x-axis in O, B, D (in order from left to right), and the ordinates at O and D meet the line $y=-\cdot 7$ in E, F, what is the area of the figure bounded by the curve OE, EF, FD?

3. Find $\int_{-1}^{+1} (x^2 - x^3)\,dx$ and interpret the result.

4. Find the area bounded by

$$y = x^3 - 6x^2 + 9x + 5$$

the x-axis and the maximum and minimum ordinates.

5. Draw the curve $y^2 = x(x-1)^2$ between $x=0$ and $x=2$ and find the area of the loop.

6. P, Q, R are the points on $y = x^3$ at which $x = 0, 2, 4$.

Find a, b, c so that the parabola

$$y = a + bx + cx^2$$

may pass through P, Q, R.

Draw the two curves on as large a scale as possible and shew that the two closed portions contained between the curves are equal in area. Also find the area of each portion.

7. P, Q, R are the points on $y = x^3$ at which $x = h-k, h, h+k$.

Find a, b, c so that the parabola

$$y = a + bx + cx^2$$

may pass through P, Q, R.

Shew that the area bounded by $y=0$, $x=h-k$, $x=h+k$ and $y=x^3$ is the same as that bounded by $y=0$, $x=h-k$, $x=h+k$ and this parabola.

8. If the area bounded by $y=x^3$, $y=0$ and any two ordinates be found by Simpson's Rule, what does (7) tell you about the result?

9. What is the area bounded by

$$y = p + qx + rx^2 + sx^3, \; y=0, \; x=-k, \; x=+k\,?$$

Notice that your result is independent of q and s.

10. If P, Q, R be the points

$$(-a, h_1) \; (0, h_2) \; (a, h_3),$$

and if the curve $y = p + qx + rx^2 + sx^3$ passes through P, Q, R, shew that p, r have the same values as in § 155 but that q, s are indeterminate.

Find the area bounded by this cubic, $y=0$, and the ordinates at P and R.

Shew that it is the same as that obtained in § 155.

11. Draw roughly $\quad y = (x+1)(2-x)$.

Find the area of the part above the x-axis.

12. Trace the curve

$$5y=(x+3)(x-1)(x-3)$$

and find the areas of the two portions bounded by the curve and the x-axis.

13. Find the area common to the two curves

$$y^2=4ax \text{ and } x^2=4ay.$$

14. Find the area bounded by $y^2=x^3$ and $x=2$.

15. P, Q are the points on the parabola $y=5+3x-2x^2$ where $x=0$ and $x=2\frac{1}{2}$. Find the area bounded by the chord PQ and the portion of the curve above it. If the tangent to the curve which is parallel to PQ meet the ordinates of P, Q in M, N, shew that the area just obtained is two-thirds of the parallelogram PQNM.

16. P, R, Q are three points on

$$y=ax^2+bx+c$$

corresponding to $\quad x=-h,\ x=0,\ x=h.$

The tangent at R meets the ordinates at P, Q in M, N. Shew that this tangent is parallel to PQ and that area PRQ bounded by the curve and the chord PQ is two-thirds of the parallelogram PQNM.

159. The fact that any definite integral may be interpreted as an area can be used to find sometimes accurately, sometimes approximately the value of a given definite integral.

Ex. Find $\quad \int_0^a \sqrt{a^2-x^2}\,dx.$

At present we do not know how to find the indefinite integral

$$\int \sqrt{a^2-x^2}\,dx.$$

But if we draw the graph of $y=\sqrt{a^2-x^2}$ between $x=0$ and $x=a$ we get a quadrant of a circle, centre at the origin, radius a. [Fig. 85.]

The given definite integral is the area of this.

$$\therefore \int_0^a \sqrt{a^2-x^2}\,dx=\frac{\pi a^2}{4}.$$

Even if we did not recognise the curve as part of a circle we could find several points on it, i.e. calculate the lengths of a number of ordinates and apply Simpson's Rule to find the approximate area.

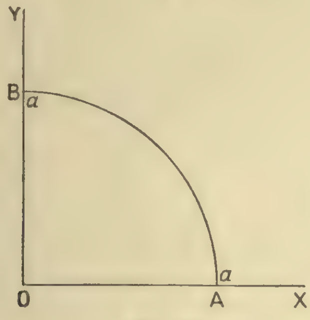


Fig. 85.

EXERCISES. XLIII.

1. Shew from a figure that

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = 2 \int_0^a \sqrt{a^2 - x^2}\, dx.$$

2. Find approximately by Simpson's rule [use 11 ordinates]

$$\int_1^2 \frac{1}{x}\, dx.$$

3. Similarly find $\int_2^3 \frac{1}{1+x^2}\, dx$ and $\int_2^3 \frac{1}{1-x^2}\, dx$.

160. If we draw $$y = \frac{1}{2}x + 2$$

and $$y = \frac{1}{4}x^2 + 2x + c$$

(which may be called the integral curve of $y = \frac{1}{2}x + 2$) between the same values of x (say 2 and 12), the number of units of length in the difference of the extreme ordinates in the second graph is

equal to the number of units of area in the figure bounded by the curve, the x-axis and the extreme ordinates in the first graph. [Fig. 86.]

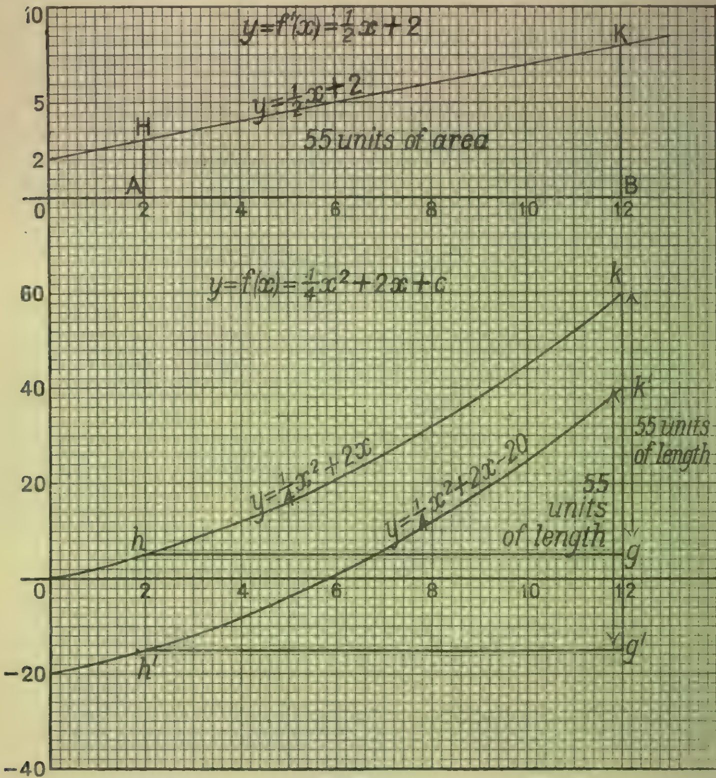


Fig. 86.

Or No. of units of length in gk (or $g'k'$) (55) = No. of units of area in AHKB and generally if the second graph be $y=f(x)$ and the first $y=f'(x)$ a corresponding result will be true.

Notice that it does not matter which curve of the family $y = \frac{1}{4}x^2 + 2x + c$ we use.

e.g. in the figure hk is $y = \frac{1}{4}x^2 + 2x \quad (c = 0)$,

$$h'k' \text{ is } y = \frac{1}{4}x^2 + 2x - 20 \quad (c = -20).$$

The ordinate of h' = (the ordinate of h) − 20
and the ordinate of k' = (the ordinate of k) − 20.

So that difference between ordinates of h' and k' = difference between ordinates of h and k.

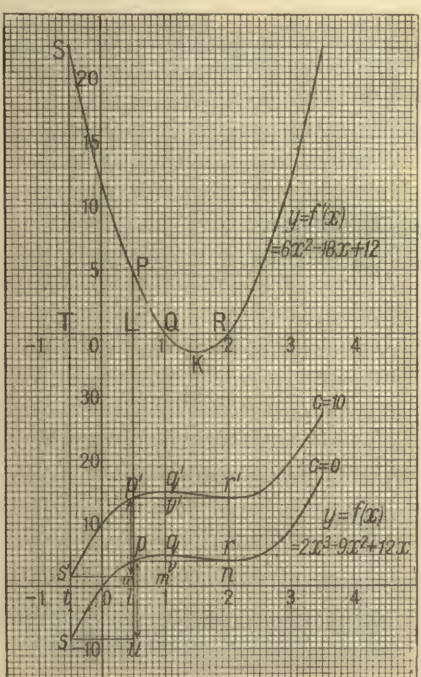


Fig. 87.

161. As another example take the graphs of

$$y = 6x^2 - 18x + 12\,[f'(x)] \quad \text{and} \quad y = 2x^3 - 9x^2 + 12x + c\,[f(x)].$$

[Fig. 87.]

No. of units of area in STLP = No. of units of length in $lp - ts$, i.e. in up or in $lp' - ts'$, i.e. in $u'p' = 12\frac{1}{2}$.

No. of units of area in PLQ = No. of units of length in $mq - lp$, i.e. in vq or in $mq' - lp'$, i.e. in $v'q' = 1$.

No. of units traced out as ordinate moves from L to R = No. of units of length in $nr - lp$ or in $nr' - lp' = 0$.

i.e. area PLQ = area QRK.

No. of units traced out as ordinate moves from Q to R = No. of units of length in $nr - mq$ or in $nr' - mq' = -vq$ or $v'q' = -1$.

i.e. area QRK = 1 unit.

162. We saw on p. 142 that if h is small

$$f(x+h) = f(x) + hf'(x)$$

to a first approximation.

We can now obtain a closer approximation.

As a first example suppose

$$f(x) = 2x^2 + 3x + 1,$$

so that

$$f'(x) = 4x + 3.$$

In this case $y = f'(x)$ is a straight line. [Fig. 88.]

Now, No. of units of length in $nq - mp$ = No. of units of area in PMNQ.

The gradient of PQ is $f''(x)$.

$$\therefore \text{RQ} = hf''(x).$$

$\therefore$ No. of units of area in PMNQ

$$= hf'(x) + \frac{1}{2}h^2 f''(x).$$

$$\therefore\ f(x+h) - f(x) = hf'(x) + \frac{1}{2}h^2 f''(x)$$

or
$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x).$$

This is accurately true in this case and in every case where $f(x)$ is of the form

$$ax^2 + bx + c,$$

since in all such cases $y = f'(x)$ is a straight line.

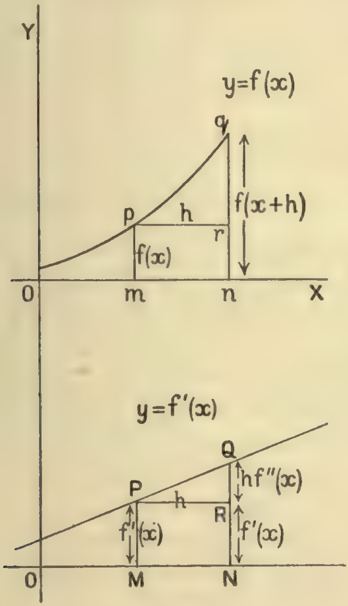


Fig. 88.

EXERCISES. XLIV.

Verify
$$f(x+h)=f(x)+hf'(x)+\frac{1}{2}h^2f'\ \ (x),$$

(i) when $f(x)=3x^2+7x-8$,

(ii) when $f(x)=ax^2+bx+c$,

(iii) when $f(x)=ax+b$.

163. If $y=f'(x)$ is not a straight line, we have as before

$f(x+h)-f(x)=$ No. of units of area in PMNQ

$=$ No. of units of area in PMNT approximately [Fig. 89]

$$=hf'(x)+\frac{1}{2}h^2f''(x).$$

i.e. $f(x+h)=f(x)+hf'(x)+\frac{1}{2}h^2f''(x)$ **approximately.**

Notice that in this case we take the arc of $y=f'(x)$ as being approximately a straight line, whereas in obtaining the first approximation we took the arc of $y=f(x)$ as being approximately straight.

e.g. taking the example on p. 143.

Find approximately the value of

$$3x^5-4x^3+6x^2-7x+8 \quad \text{when} \quad x=3{\cdot}01.$$

$$f'(x)=15x^4-12x^2+12x-7,$$

$$f''(x)=60x^3-24x+12.$$

$$\therefore\ f(3)=662, \quad f'(3)=1136, \quad f''(3)=1560,$$

$$\therefore\ f(3{\cdot}01)=f(3)+{\cdot}01\,f'(3)+\frac{({\cdot}01)^2}{2}f''(3) \text{ app.}$$

$$=662+1136\times{\cdot}01+\frac{1560}{2}\times{\cdot}0001$$

$$=673{\cdot}4380 \text{ app.}$$

[Actually $f(3{\cdot}01)=673{\cdot}438264503.$]

164. Mean or average ordinate. Suppose AB to be a portion of the curve $y=f(x)$, A, B being the points corresponding to $x=a$ and $x=b$. [Fig. 90.]

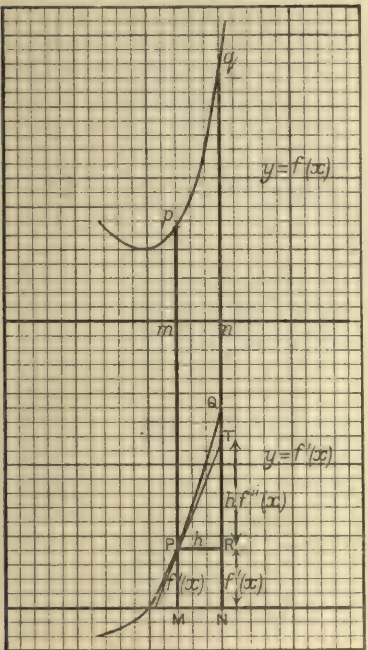


Fig. 89.

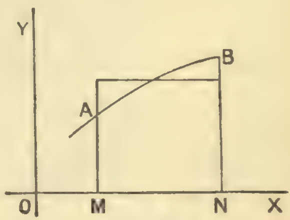


Fig. 90.

The mean ordinate of the curve between $x=a$ and $x=b$ is the height of the rectangle with base MN equal in area to the figure AMNB.

It is in fact that ordinate of constant length which in moving from M to N traces out the same area that the variable ordinate of the curve does in moving from M to N.

[Compare idea of average speed p. 2.]

165. Suppose for example, we want the mean ordinate of the curve $y = x^2$ between $x = 1$ and $x = 5$. [Fig. 91.]

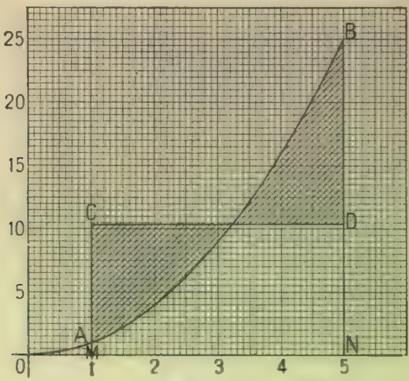


Fig. 91.

$$\text{Area AMNB} = \int_1^5 x^2\,dx = \left[\frac{x^3}{3}\right]_1^5 = \frac{124}{3},$$

and $$\text{MN} = 4.$$

$$\therefore \text{ Mean ordinate} = \frac{124}{3 \times 4} = \frac{31}{3} = 10\tfrac{1}{3}.$$

If CD be the line $y = 10\frac{1}{3}$,

$$\text{Area CMND} = \text{area AMNB}.$$

EXERCISES. XLV.

1. Find the mean ordinate of the curve $y = x^2$ between $x = 4$ and $x = 10$.

Find also the arithmetic mean of the two ordinates $x = 4$, $x = 10$ and the midway ordinate corresponding to $x = 7$.

2. Find the mean ordinate of the line $y = 3x + 5$ between $x = 3$ and $x = 15$.

Find also the arithmetic mean of the two ordinates $x = 3$ and $x = 15$ and the midway ordinate.

3. Prove by calculus and geometrically that the mean ordinate of

$$y=mx+n \text{ between } x=a \text{ and } x=b,$$

the arithmetic mean of the ordinates $x=a$, $x=b$ and the midway ordinate are all equal.

4. Find the mean ordinate of $y=x^3$ (i) between $x=3$ and $x=7$, (ii) between $x=a$ and $x=b$.

5. Find the mean ordinate of $y=x^2-4x+3$ between $x=1$ and $x=5$.

6. Find the mean ordinate of $y=x^2$ between $x=0$ and $x=+3$, also between $x=-3$ and $x=+3$ and explain why these results are the same.

166. If we divide the area into a number of strips (say 20) of equal width ($\cdot 2$), and complete the inside rectangles in the usual way with an extra one on the right [Fig. 92], the area of

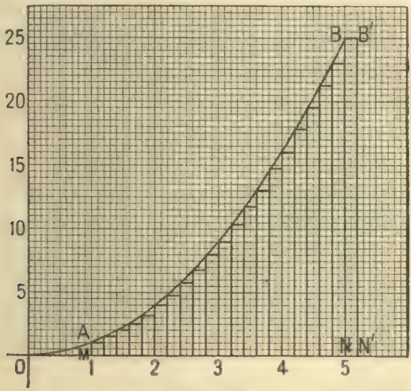


Fig. 92.

the figure bounded by AM, MN′, N′B′ and the zig-zag boundary AB′ is

$$\cdot 2\,[1^2+1\cdot 2^2+1\cdot 4^2+\ldots+5^2]$$

and $$\text{MN}'=4\cdot 2.$$

So that the mean height of this figure is

$$\frac{\cdot 2\,[1^2+1\cdot 2^2+\ldots+5^2]}{4\cdot 2}=\frac{1^2+1\cdot 2^2+\ldots+5^2}{21}$$

= the Arithmetic mean of the 21 ordinates from AM to BN inclusive.

[This is $10\frac{7}{15} = 10{\cdot}47$.]

If we take 100 strips the Arithmetic mean of the 101 ordinates is 10·43.

The more strips we take the nearer does the Arithmetic mean approach to $10\frac{1}{3}$.

This is only what might be expected, for we have shewn that by increasing the number of strips the area AMN′B′ can be brought as near to the area bounded by AM, MN, NB and the curve BA as we please.

The value $10\frac{1}{3}$ may thus be called the mean or average value of x^2 between $x=1$ and $x=5$, and it is the limit to which the Arithmetic mean of a number of equidistant ordinates the first of which is AM and the last BN, approaches as the number of ordinates is indefinitely increased.

EXERCISES. XLVI.

1. Find the mean value of x^2+3x between $x=2$ and $x=4$.

Find also the Arithmetic mean of 11 values of x^2+3x corresponding to

$$x=2,\ 2{\cdot}2,\ 2{\cdot}4,\ 2{\cdot}6,\ \dots 4.$$

2. Find the mean value of $\dfrac{1}{x^2}$ between $x=1$ and $x=4$.

3. Use Simpson's rule to find approximately the mean value of $\dfrac{1}{x}$ between $x=1$ and $x=2$. (11 ordinates.)

Also use a table of reciprocals to find the Arithmetic mean of 11 values of $\dfrac{1}{x}$ corresponding to $x=1,\ 1{\cdot}1,\ 1{\cdot}2,\ \dots 2$.

4. Find the mean value of x^3+2x^2+7 between $x=2$ and $x=7$.

167. If part of the curve $y=f(x)$ is below the x-axis the corresponding portion of area must be reckoned negative.

Thus to get the mean ordinate of

$$y=x^3$$

between $$x=-2 \text{ and } x=3.$$

We have

$$\int_{-2}^{3} x^3\,dx = \left[\frac{x^4}{4}\right]_{-2}^{3} = \frac{65}{4},$$

and this gives actual area ONB – actual area OMA.

$$\therefore \text{ Mean ordinate} = \frac{65}{4 \times 5} = \frac{13}{4} = 3\tfrac{1}{4}.$$

If CD be $$y = \frac{13}{4},$$

Area CMND = actual area ONB – actual area OMA [Fig. 93]

or = algebraic sum of areas ONB, OMA.

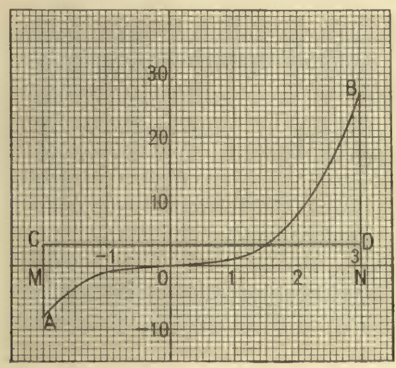


Fig. 93.

If we take 26 equidistant ordinates—distance apart ·2—the Arithmetic mean will be

$$\frac{(-2)^3 + (-1{\cdot}8)^3 + (-1{\cdot}6)^3 + \ldots (-{\cdot}2)^3 + 0^3 + ({\cdot}2)^3 + ({\cdot}4)^3 + \ldots + 3^3}{26}$$

$$= \frac{2{\cdot}2^3 + 2{\cdot}4^3 + \ldots 3^3}{26} = \frac{91}{26} = 3\tfrac{1}{2}$$

and as before the greater the number of ordinates we take the nearer will the Arithmetic mean be to $3\tfrac{1}{4}$.

EXERCISES. XLVII.

1. Find the mean ordinate of $y=2x+6$ between $x=-11$ and $x=5$. Draw a figure to explain the result.

2. Find the mean ordinate of $y=x^3$

(i) between 0 and 3,

(ii) between -3 and 3.

Draw a figure.

3. Find the mean value of $4+3x-x^2$ between $x=-2$ and $x=3$. Find also the Arithmetic mean of 11 equidistant ordinates.

4. Find the mean height with respect to x of that portion of

$$y=4+3x-x^2$$

which lies above the x-axis.

5. Find the mean value with respect to the abscissa of the square of the ordinate of a semicircle of radius a.

6. Find the mean value of $2x^3-3x^2-36x+30$ between $x=-3$ and $x=4$.

CHAPTER VIII

FURTHER APPLICATIONS OF THE INTEGRAL CALCULUS

168. So far we have only considered the Integral Calculus in its application to the areas of plane curves.

We shall now shew how similar methods may be applied to other problems.

Ex. 1. The speed of a body in ft./sec. at the end of t seconds is given by

$$\mathbf{v = 3t^2.}$$

Find the distance travelled in 12 seconds from rest.

Suppose the whole time 12 secs. to be divided into n equal intervals each h secs., so that $nh = 12$, and suppose the speed to remain constant during each interval and equal to the speed at the beginning of the interval.

The speeds at the beginning of successive intervals are

$$0,\ 3h^2,\ 3\,(2h)^2,\ \dots\ 3\,(\overline{n-1}h)^2,$$

and the total distance described on our assumption would be

$$h\,[0 + 3h^2 + 3\,.\,(2h)^2 + \dots + 3\,(\overline{n-1}h)^2]$$

$$= 3h^3\,[1^2 + 2^2 + 3^2 + \dots\ (n-1) \text{ terms}] = \frac{3h^3}{6}\,(n-1)\,n\,(2n-1)$$

$$= \frac{h^3}{2}\,[2n^3 - 3n^2 + n] = \frac{h^3}{2}\left[\frac{2 \times 1728}{h^3} - \frac{3 \times 144}{h^2} + \frac{12}{h}\right]$$

$$\mathbf{= 1728 - 216h + 6h^2}\ (= \mathrm{s}_1).$$

Now suppose the speed to remain constant during each interval and equal to the speed at the end of the interval. The total distance would be

$$h\left[3h^2 + 3 \,.\, (2h)^2 + \ldots + 3\,(nh)^2\right]$$
$$= 3h^3\left[1^2 + 2^2 + 3^2 + \ldots\, n \text{ terms}\right]$$
$$= \frac{3h^3}{6} \,.\, n\,(n+1)\,(2n+1)$$
$$= \mathbf{1728 + 216h + 6h^2}\ (= s_2).$$

Now the actual distance (s) lies between s_1 and s_2. But by diminishing h indefinitely we can make s_1 and s_2 each as near to 1728 as we like.

$\therefore$ s is the limit of either s_1 or s_2 as $h \to 0$, and $s = 1728$.

Notice that it is unnecessary to find both s_1 and s_2. We can say

either $\quad s_1 = 1728 - 216h + 6h^2, \quad \therefore\ s = 1728,$

or $\quad s_2 = 1728 + 216h + 6h^2, \quad \therefore\ s = 1728.$

EXERCISES. XLVIII.

If $v = 5 + 7t$ find by this method the distance travelled,

(i) in the first 5 seconds,

(ii) in the 10th second.

Also in each case find what the distance would be if we supposed the time to be divided up into intervals of ·01 sec. and the speed to remain constant during each interval and equal to the speed at the beginning of the interval.

Find by how much per cent. the distance calculated on this assumption differs from the actual distance.

169. The method employed in § 168 corresponds exactly to that employed in the determination of an area in § 138. There we had a continually changing ordinate (y) whose length was given as a function of the abscissa (x). Here we have a continually changing speed (v) whose magnitude is given as a function of the time (t). There we got an approximation to the area traced out

by the moving ordinate by dividing the distance travelled by it into small pieces and supposing the ordinate to remain the same length as it passed from end to end of each small piece: we then added the areas of the thin rectangles so obtained and finally found the limit of the sum of these areas as the number of them was indefinitely increased and the breadth of each consequently indefinitely diminished.

Here we get an approximation to the distance travelled by the body by dividing the whole time into small intervals and supposing the speed to remain constant during each interval; we then add the small distances so obtained and finally find the limit of the sum of these distances, as the number of them is indefinitely increased and the length of each interval consequently indefinitely diminished.

170. We shall now approach the problem from a different point of view corresponding to that adopted in § 140, in finding the area traced out by a moving ordinate.

Let s ft. be the distance described in t secs. and $s+\Delta s$ ft. be the distance described in $t+\Delta t$ secs.

The speed at the beginning of the interval Δt is $3t^2$ ft./sec. and at the end $3(t+\Delta t)^2$.

$\therefore$ Δs lies between $3t^2 . \Delta t$ and $3(t+\Delta t)^2 \Delta t$,

$$\therefore \frac{\Delta s}{\Delta t} \text{ lies between } 3t^2 \text{ and } 3(t+\Delta t)^2.$$

But as $\Delta t \to 0$, $(t+\Delta t)^2 \to t^2$.

$$\therefore \; {}^*\frac{ds}{dt} = 3t^2.$$

$\therefore$ $s = t^3 + c$ where c is some constant.

* [So far as finding $\frac{ds}{dt}$ is concerned we might simply say

$$\Delta s = 3t^2 . \Delta t \text{ app.}$$

$$\therefore \frac{ds}{dt} = 3t^2. \quad \text{v. § 147.]}$$

Since we are finding the distance the body has travelled from its position when $t=0$, we have $s=0$ when $t=0$,

$$\therefore\ 0=0+c \text{ or } c=0.$$

$\therefore$ Distance in 12 secs. $=12^3=$ **1728** ft.

Notice that this is $\Big[t^3\Big]_0^{12}$.

EXERCISES. XLIX.

1. Find by this method the distance travelled in the 5th second when $v=3t^2$ [ft.-sec. units].

2. Find the distance travelled between the ends of the 3rd and 10th seconds when $v=2t^3+3t+5$.

171. Generally if $v=f(t)$ and we want the distance between the end of a secs. and the end of b secs., we say Δs lies between $f(t)\,.\,\Delta t$ and $f(t+\Delta t)\,.\,\Delta t$ and as before we deduce

$$\frac{ds}{dt}=f(t) \text{ or } v.$$

Let $\phi(t)$ be a function of t which when differentiated gives $f(t)$.

$$\therefore\ s=\phi(t)+c.$$

Now $s=0$ when $t=a$.

$$\therefore\ c=-\phi(a),$$

i.e.
$$s=\phi(t)-\phi(a),$$

and the distance required $=\phi(b)-\phi(a)=\Big[\phi(t)\Big]_a^b$.

172. Each term in the first series in § 168 is the distance travelled in a time h, the speed being supposed constant throughout the interval. The series is

$$h\,[0+3h^2+3\,.\,(2h)^2+\ldots],$$

(0, $3h^2$, $3(2h)^2 \dots$ being the values of v or $3t^2$ when $t=0$, h, $2h$, ...),

or shortly $$\sum_{t=0}^{t=12-h} 3t^2 . h \quad \text{or} \quad \sum_{t=0}^{t=12-h} v . h.$$

As in § 148 we may shew that the limits of $\sum_{t=0}^{t=12-h} 3t^2 . h$ and $\sum_{t=0}^{t=12} 3t^2 . h$ when $h \to 0$ are the same.

$\therefore$ we may say that the actual distance required is the limit of $\sum_{t=0}^{t=12} 3t^2 . h$ when h is indefinitely diminished.

173. Generally if $v = f(t)$, the distance described in a small interval Δt following the end of t seconds is approximately $v . \Delta t$ and the sum of all such distances between the end of a seconds and the end of b seconds may be denoted approximately by

$$\sum_{t=a}^{t=b} v \Delta t.$$

The actual distance is the limit of this when $\Delta t \to 0$ and this is denoted by

$$\int_a^b v\,dt \quad \text{or} \quad \int_a^b f(t) . dt,$$

and we have seen that this is $\phi(b) - \phi(a)$ or $\Big[\phi(t)\Big]_a^b$

where $$\frac{d\phi(t)}{dt} = f(t).$$

e.g. In our problem (§§ 168, 170)

$$\text{distance} = \int_0^{12} 3t^2 . dt = \Big[t^3\Big]_0^{12} = 1728.$$

174. This problem can be reduced to the problem of finding an area as follows:

The relation $v = 3t^2$ can be exhibited in the form of a graph. [Fig. 94.]

Distances along the horizontal axis represent seconds of time and distances along the vertical axis represent speed in ft./sec.

OM represents t seconds, i.e. OM contains t horizontal units of length.

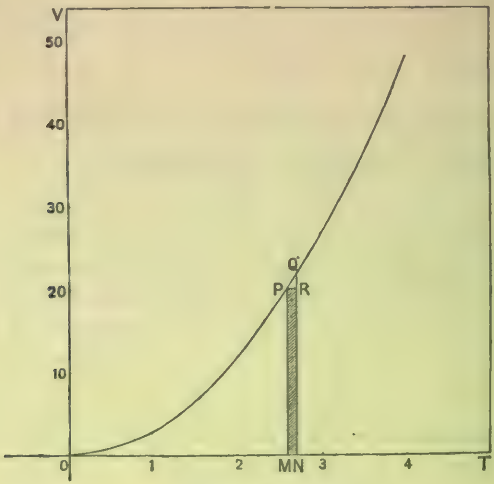


Fig. 94.

MP represents a speed of $3t^2$ ft./sec., i.e. MP contains $3t^2$ vertical units of length.

ON represents $(t+\Delta t)$ seconds and NQ $3(t+\Delta t)^2$ ft./sec.

The number of feet described in the interval Δt on our first hypothesis is $v\,\Delta t$, and this is the number of units of area in the rectangle PMNR.

The number of feet described in 12 seconds will be the number of units of area in all strips like PMNR between O and the ordinate corresponding to $t=12$.

In other words $\sum\limits_{t=0}^{t=12} v\Delta t$ may be looked upon either as the number of feet described in all the intervals or as the number of

units of area in all the strips and the limit of this when $\Delta t \to 0$ (which we call $\int_0^{12} v\,dt$) is either the number of feet described in 12 seconds or the number of units of area in the curvilinear figure OAB where AB is the ordinate $x = 12$.

General statement for speed-time graph.

175. Generally. If the speed be given as a function of the time, say $v = f(t)$, and the speed-time graph be drawn, the

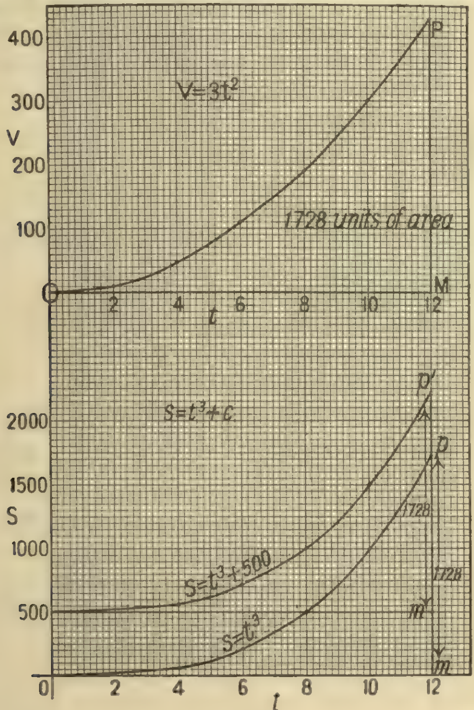


Fig. 95.

number of units of area bounded by the curve, $v=0$, $t=a$, $t=b$, gives the number of units of length in the distance described between the end of a seconds and the end of b seconds.

If the speed-time and space-time graphs be drawn, the number of units of length in the difference between two ordinates of the space-time graph is equal to the number of units of area in the corresponding portion of the speed-time graph.

e.g. number of units of area in OPM is number of units of length in mp or $m'p'$. [Fig. 95.]

i.e. the number of units of area lying between the speed-time curve, the time-axis and the ordinates $t=a$, $t=b$ is the number of units of length in the distance travelled between the end of a seconds and the end of b seconds.

Another example.

176. If $$v=112-32t,$$

$$\int_2^5 v\,dt=\left[112t-16t^2\right]_2^5=112\times 3-16\times 21=0.$$

The meaning of this is not hard to find.

From $t=2$ to $t=3\frac{1}{2}$, v is positive.

From $t=3\frac{1}{2}$ to $t=5$, v is negative.

So that of the terms which go to make up $\sum\limits_{t=2}^{t=5} v\Delta t$, some are positive and some negative and our result tells us that the sums of positive and negative terms are numerically equal.

As a matter of fact our formula gives the speed at any time of a body projected vertically with speed 112 ft./sec.

From $t=2$ to $t=3\frac{1}{2}$ the body ascends and from $t=3\frac{1}{2}$ to $t=5$ it descends an equal distance, so that its distance from the starting point, or what we may call the effective distance described, is zero.

If we want the actual distance described we must integrate from 2 to $3\frac{1}{2}$.

If we draw the graph $v = 112 - 32t$ we get a straight line cutting $y = 0$ where $x = 3\frac{1}{2}$. From $x = 2$ to $x = 3\frac{1}{2}$ the area is above the x-axis and from $x = 3\frac{1}{2}$ to $x = 5$ below and the two portions of area are numerically equal but of opposite sign. [Fig. 96.]

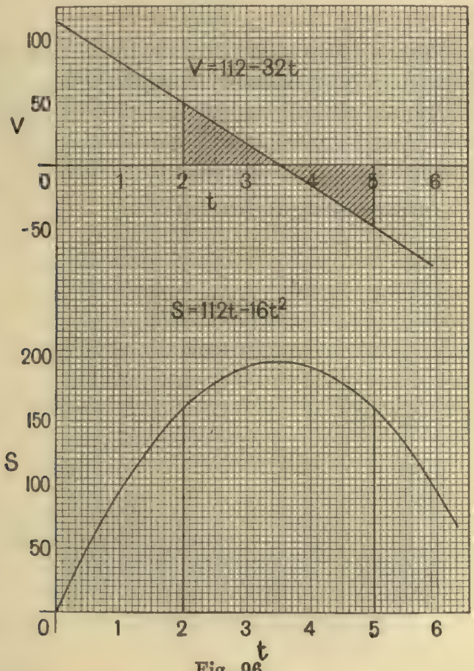


Fig. 96.

The corresponding ordinates in the space-time graph are equal.

177. *Ex.* 2. Find the work done in stretching an elastic string from a length of 18 to a length of 25 inches, given that the natural length is 8 inches and that it has a length of 14 inches when sustaining a pull of 3 lbs. wt.

By Hooke's law if the pull be T lbs. wt. and the stretch s inches,

$$T = ms \text{ (where } m \text{ is a constant).}$$

Now when $T = 3$, $s = 6$,

$$\therefore \ T = \frac{1}{2} s.$$

Divide the total extension of 7 inches into n equal parts, each h inches, so that $nh = 7$ and suppose that as the string was stretched from 18 to $(18+h)$ inches the pull remained the same as it was at 18 inches and that as it was stretched from $(18+h)$ to $(18+2h)$ the pull remained the same as at $(18+h)$, and so on.

The pulls corresponding to lengths

$$18, \ 18+h, \ 18+2h, \dots 25-h,$$

are

$$5, \quad 5+\frac{h}{2}, \quad 5+h, \quad \dots \frac{17-h}{2},$$

and the work done on our assumption would be

$$h\left[5+\left(5+\frac{h}{2}\right)+(5+h)+\dots\frac{17-h}{2}\right] \text{ in. lbs. wt.}$$

$$= \frac{nh}{2} \cdot \frac{27-h}{2}$$

$$= \frac{27}{4} \cdot nh - \frac{1}{4} \cdot nh \cdot h = \frac{27}{4} \cdot 7 - \frac{7}{4} h = \left(\frac{189}{4} - \frac{7h}{4}\right) \text{ in. lbs. wt.} = W_1.$$

If we supposed that as the string was stretched from 18 to $(18+h)$ inches the pull was the same as at $18+h$, and so on, we should get as the work done

$$\left(\frac{189}{4} + \frac{7h}{4}\right) \text{ in. lbs. wt.} = W_2.$$

Now the actual work done lies between W_1 and W_2 but each of these can be made as near $\frac{189}{4}$ as we like if h be made small enough.

$\therefore$ Actual work (W in. lbs. wt.) is the limit of either W_1 or W_2 when $h \to 0$ and is $47\frac{1}{4}$ in. lbs. wt.

Notice that we need not calculate both W_1 and W_2. We can say

either $$\mathsf{W}_1 = \frac{189}{4} - \frac{7h}{4}, \quad \therefore \mathsf{W} = \frac{189}{4},$$

or $$\mathsf{W}_2 = \frac{189}{4} + \frac{7h}{4}, \quad \therefore \mathsf{W} = \frac{189}{4}.$$

178. The method employed here corresponds to that employed in §§ 138 and 168. We shall now solve the problem by a method corresponding to that employed in §§ 140 and 170.

179. Let W in. lbs. wt. be the work done in stretching the string from 18 ins. to x ins. and $(\mathsf{W} + \Delta\mathsf{W})$ in. lbs. wt. the work done in stretching it from 18 ins. to $(x + \Delta x)$ ins.

$$\left.\begin{array}{l} \text{The pull corresponding to length } x \text{ ins. is } \dfrac{x-8}{2} \text{ lbs. wt.} \\ \text{The pull corresponding to length } (x+\Delta x) \text{ ins. is} \\ \qquad\qquad \dfrac{x+\Delta x - 8}{2} \text{ lbs. wt.} \end{array}\right\}.$$

$\therefore$ the work done in stretching from x ins. to $(x + \Delta x)$ ins. lies between $\dfrac{x-8}{2}\Delta x$ and $\dfrac{x+\Delta x-8}{2}\Delta x$ in. lbs. wt.

i.e. $\Delta\mathsf{W}$ lies between $\dfrac{x-8}{2}\Delta x$ and $\dfrac{x+\Delta x-8}{2}\Delta x$ in. lbs. wt.

$\therefore$ $\dfrac{\Delta\mathsf{W}}{\Delta x}$ lies between $\dfrac{x-8}{2}$ and $\dfrac{x+\Delta x-8}{2}$ in. lbs. wt.

$$\therefore \boldsymbol{\frac{d\mathsf{W}}{dx} = \frac{x-8}{2} = \frac{x}{2} - 4}\text{*}.$$

$$\therefore \mathsf{W} = \frac{x^2}{4} - 4x + c.$$

* So far as finding $\dfrac{d\mathsf{W}}{dx}$ is concerned it is sufficient to say

$$\Delta\mathsf{W} = \frac{x-8}{2}\Delta x \text{ approximately.}$$

$$\therefore \frac{\Delta\mathsf{W}}{\Delta x} = \frac{x-8}{2} \text{ approximately.}$$

$$\therefore \frac{d\mathsf{W}}{dx} = \frac{x-8}{2}. \quad \text{[v. § 147.]}$$

Now when $x=18$, $W=0$.

$$\therefore\ 0=\frac{18^2}{4}-4.18+c,$$

$$\therefore\ c=-9,$$

$$\therefore\ \mathbf{W=\frac{x^2}{4}-4x-9.}$$

$$\therefore \text{ work done in stretching to 25 inches}=\frac{625}{4}-100-9$$

$$=56\tfrac{1}{4}-9$$

$$=47\tfrac{1}{4} \text{ in. lbs. wt.}$$

Notice that $56\frac{1}{4}$ and 9 are the values of $\frac{x^2}{4}-4x$ when $x=25$ and 18 respectively, and

$$\text{work done}=\left[\frac{x^2}{4}-4x\right]_{18}^{25}.$$

180. Each term in the series in § 177 is the work done in stretching the string through a small distance h, the pull being supposed to remain constant throughout this small stretch.

The series is $\quad h\left[5+\left(5+\frac{h}{2}\right)+(5+h)+\ldots\right]$ in. lbs. wt.

(5, $5+\frac{h}{2}$, $5+h$ ... being the values of T or $\frac{x-8}{2}$, when $x=18$, $18+h$, $18+2h$ ...).

Or shortly $\quad \sum\limits_{x=18}^{x=25-h} h\,.\,\frac{x-8}{2}$ or $\sum\limits_{x=18}^{x=25-h} hT$.

As in § 148 we may shew that the limits of $\sum\limits_{x=18}^{x=25-h} hT$ and $\sum\limits_{x=18}^{x=25} hT$ when $h\to 0$ are the same.

$\therefore$ we may say that the actual distance required is the limit of $\sum\limits_{x=18}^{x=25} hT$ when $h\to 0$.

181. Generally if $T=f(x)$ gives the pull in terms of the length, the work done in stretching from length x to length $(x+\Delta x)$ is approximately $T\,.\,\Delta x$ and the sum of all such works done as the length increases from a to b may be denoted approximately by

$$\sum_{x=a}^{x=b} T\,.\,\Delta x$$

The actual work done is the limit of this when $\Delta x \to 0$ and this is denoted by

$$\int_a^b T\,dx \quad \text{or} \quad \int_a^b f(x)\,.\,dx,$$

and we have seen that this is $\phi(b)-\phi(a)$ or $\Big[\phi(x)\Big]_a^b$ where

$$\frac{d\phi(x)}{dx}=f(x).$$

e.g. In our problem

$$\text{work done} = \int_{18}^{25}\frac{x-8}{2}\,.\,dx = \left[\frac{x^2}{4}-4x\right]_{18}^{25} = 47\tfrac{1}{4} \text{ in. lbs. wt.}$$

182. This can also be reduced to finding the area of a curve as follows.

The work done is $\int_{18}^{25} T\,dx$ where T is a given function of x, in this case $\frac{x-8}{2}$.

This relation $T=\frac{x-8}{2}$ can be exhibited in the form of a graph. [Fig. 97.]

Distances along the horizontal axis represent length of string in ins., and distances along the vertical axis represent pull of string in lbs. wt.

In this case the graph is a straight line.

OM represents x ins., i.e. OM contains x horizontal units, MP represents a pull of $T\left(=\frac{x-8}{2}\right)$ lbs. wt., i.e. MP contains $T\left(=\frac{x-8}{2}\right)$ vertical units.

ON represents $(x + \Delta x)$ ins. and NQ a pull of

$$\mathsf{T} + \Delta\mathsf{T}\left(=\frac{x + \Delta x - 8}{2}\right) \text{ lbs. wt.}$$

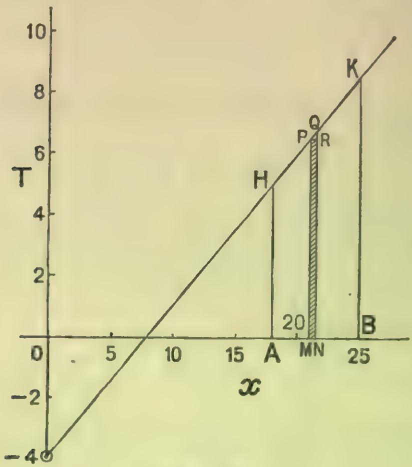


Fig. 97.

The work done on our first hypothesis as the string is stretched from length x to length $x + \Delta x$ is $\mathsf{T} \, . \, \Delta x$ in. lbs. wt. and this is the number of units of area in the rectangle PMNR. The work done in stretching the string from 18 to 25 ins. will be the number of units of area in all the strips like PMNR between A and B.

In other words $\sum\limits_{x=18}^{x=25} \mathsf{T} \, . \, \Delta x$ may be looked upon either as the number of in. lbs. wt. of work done on our first assumption or as the number of units of area in all strips like PMNR, and the limit to which this tends as $\Delta x \to 0$ which we call $\int_{18}^{25} \mathsf{T} \, dx$ is either the number of in. lbs. wt. of work done or the number of units of area in the figure ABKH.

183. The following statements should now be intelligible.

(1) **Integration is a process of summation.**

If $f(x)$ be any function of x, the values which $f(x)$ takes as x increases from a to b by equal increments h are

$$f(a),\ f(a+h),\ f(a+2h)\ \dots.$$

The limit of the sum $h \,.\, f(a) + hf(a+h) + \dots hf(b)$ when $h \to 0$ or as it may be briefly written the limit of $\sum\limits_{x=a}^{x=b} f(x) \,.\, \Delta x$ when $\Delta x \to 0$ is written $\int_a^b f(x) \,.\, dx$ and the finding of this limit is called integration.

(2) **Integration is a process of anti-differentiation.**

To find the limit of the sum just mentioned it is necessary to discover a function of x, say $\phi(x)$, such that $\dfrac{d\phi(x)}{dx} = f(x)$, i.e. to find a function $\phi(x)$ which when differentiated gives $f(x)$.

This process of finding $\phi(x)$ which may be called a process of anti-differentiation is also called integration.

If $$\frac{d\phi(x)}{dx} = f(x), \quad \int f(x) \,.\, dx = \phi(x) + c.$$

(3) **Any definite integral may be interpreted as an area.**

For $\int_a^b f(x) \,.\, dx$ is the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$.

Therefore, even if we cannot find the value of the definite integral by the ordinary methods, that is to say if we cannot discover a function $\phi(x)$ such that $\dfrac{d\phi(x)}{dx} = f(x)$, we can find its value approximately by drawing the curve $y = f(x)$ between $x = a$

and $x=b$ and applying some one of the approximate methods previously mentioned for finding the area.

184. In any problem requiring the Integral Calculus for its solution it is advisable to keep in mind what we should do if we wanted an approximate solution.

Ex. 1. Find the area bounded by $y=x^3$, $y=0$, $x=3$, $x=7$.

We have to find the area traced out by an ordinate which moves from the position AH to the position BK, its length changing continually in accordance with the law $y=x^3$. [Fig. 98.]

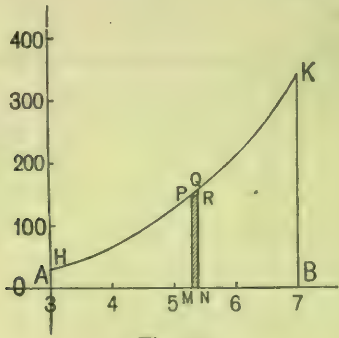


Fig. 98.

To get an approximate area we should divide AB into a number of equal parts like MN, and suppose the ordinate to keep the same length MP as it moved from M to N.

The sum of all such rectangles as MR would be an approximation to the area required, and we have seen that the actual area required is the limit of the sum of such rectangles when their width is indefinitely diminished.

We have seen that so far as the value of this limit is concerned it is immaterial whether we suppose the ordinate in its passage

from M to N to preserve its initial length MP or its final length NQ, or of course any intermediate length.

The work may be stated thus :

Suppose the area divided into a number of strips by lines parallel to OY.

Let PMNQ be a typical strip, MP being the ordinate corresponding to abscissa x. $\therefore$ MP $= x^3$ and area PMNQ $= x^3 \Delta x$ approximately and sum of strips $= \sum_{x=3}^{x=7} x^3 \Delta x$ approximately.

The actual area required $= \int_3^7 x^3 dx$

$$= \left[\frac{x^4}{4}\right]_3^7 = \frac{7^4 - 3^4}{4} = 580.$$

Ex. 2. The speed (v ft./sec.) of a body at the end of t seconds is given by

$$\mathbf{v = 6t + 17t^2}.$$

Find the distance travelled between the end of the 2nd and the end of the 5th second.

Here we have to find the distance travelled in a certain time by a body whose speed continually changes in accordance with the law $\mathbf{v = 6t + 17t^2}$.

To get an approximate result we should divide the whole time into a number of small intervals and suppose the speed to keep throughout each interval the value which it has at the beginning of the interval. We should then calculate the distance travelled in each interval on this assumption and the sum of all the distances would be an approximation to the required distance.

The actual distance is the limit of this sum when the duration of each interval is indefinitely diminished. We have seen that so far as the value of this limit is concerned it is immaterial whether we suppose the speed throughout an interval to remain

the same as it was at the beginning or at the end of the interval, or indeed at any intermediate instant.

The work may be stated thus :

Suppose the whole time divided into a number of small intervals.

The interval Δt following the end of t seconds may be taken as the typical interval.

The speed at the beginning of this interval is

$$(6t + 17t^2) \text{ ft./sec.}$$

$\therefore$ Distance travelled in this interval $= (6t + 17t^2)\,\Delta t$ feet approximately and the total distance is $\sum\limits_{t=2}^{t=5} (6t + 17t^2)\,\Delta t$ approximately.

Actually, distance is $\displaystyle\int_2^5 (6t + 17t^2)\,dt$

$$= \left[3t^2 + \frac{17}{3}t^3\right]_2^5$$

$$= 3(5^2 - 2^2) + \frac{17}{3}(5^3 - 2^3)$$

$$= 63 + 663$$

$$= 726 \text{ feet.}$$

Ex. 3. A tank 6 feet deep is filled with water. Find the magnitude of the resultant thrust on an end which is 4 feet wide.

The pressure at any point is proportional to the depth and is therefore continually changing as we pass downwards from top to bottom of the tank.

We should get an approximation to the required thrust by dividing the area into a number of thin strips by horizontal lines, and supposing the pressure to remain constant over each strip. If PQNM is such a strip we suppose the pressure not to change as we pass from the level PM to the level QN but to keep throughout the value that it has at the level PM. [Fig. 99.]

We should then calculate the thrust on the strip PQNM on this assumption, and the sum of the thrusts on all the strips would be an approximation to the required result.

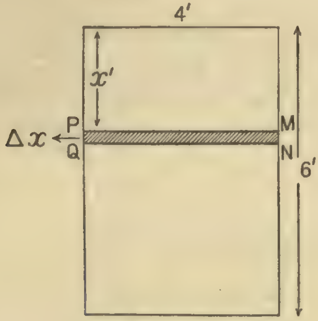


Fig. 99.

The actual resultant thrust is the limit of this sum when the width of each strip is indefinitely diminished. The limit would be the same if we supposed the pressure at every point of PQNM to be the same as it is at the level QN or at any level intermediate between PM and QN.

The work may be stated thus:

Suppose the end divided into thin strips by horizontal lines.

PQNM is such a strip, the depth of P being x feet and $PQ = \Delta x$ feet.

The pressure at the level PM = wt. of x cubic feet of water per sq. foot $= 62{\cdot}5x$ lbs. wt./sq. ft.

The area of the strip $= 4\Delta x$ sq. ft.

$\therefore$ Thrust on strip $= 62{\cdot}5x \times 4\Delta x$ lbs. wt. approximately

$= 250x\Delta x$ lbs. wt. approximately.

$\therefore$ Total thrust on end $= \sum\limits_{x=0}^{x=6} 250x\Delta x$ lbs. wt. approximately.

Actually,
$$\text{thrust} = \int_0^6 250x\,dx$$
$$= 250\left[\frac{x^2}{2}\right]_0^6$$
$$= \mathbf{4500\ lbs.\ wt.}$$

Ex. 4. Find the work done by a gas in expanding from 2 to 3 cubic feet, the pressure and volume being connected by the law $pv =$ constant, given that the pressure is 2160 lbs. wt./sq. ft. when the volume is $\frac{1}{4}$ cubic foot.

$$\left.\begin{array}{l}\text{Let } p \text{ lbs. wt./sq. ft. be the pressure}\\ \quad v \text{ cub. ft. the volume}\end{array}\right\}.$$

$$\therefore pv = 2160 \times \frac{1}{4} = 540.$$

Suppose the gas contained in a cylinder closed at one end and fitted with a piston free to slide along the cylinder.

Call the area of the piston a sq. ft. [Fig. 100.]

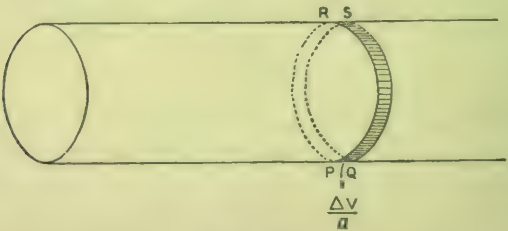


Fig. 100.

When the volume of the gas increases from v to $v + \Delta v$ the piston moves $\frac{\Delta v}{a}$ feet and the force on the piston at the beginning of this small movement is $\frac{540}{v} \cdot a$ lbs. wt.

∴ Work done by gas when volume increases from v to $v+\Delta v$ is

$$\frac{540}{v} \cdot a \cdot \frac{\Delta v}{a} \text{ ft.-lbs. wt. approximately}$$

$$= \frac{540}{v} \cdot \Delta v \text{ ft.-lbs. wt. approximately,}$$

and the total work done in expanding from 2 to 3 cubic feet is

$$\sum_{v=2}^{v=3} \frac{540}{v} \Delta v \text{ app.,}$$

actually it is
$$\int_2^3 \frac{540}{v} \cdot dv.$$

At present we do not know how to find the value of the indefinite integral $\int \frac{1}{v} dv$, but we can obtain an approximate value for the definite integral by looking upon it as an area and applying Simpson's rule. [Fig. 101.]

Taking 11 ordinates, we have

Extreme	*Even*	*Odd*
270	257·14	245·45
180	234·78	225
	216	207·69
	200	192·86
	186·21	
450	1094·13	871·00
	4376·52	
	1742·00	
	450	
	6568·52	

$$\text{Area} \doteqdot \frac{\cdot 1}{3} \times 6568{\cdot}5$$

$$= 219 \text{ nearly.}$$

∴ Work done = 219 ft.-lbs. wt.

[We shall see later, Exs. LXXIX. 6, that this is correct to three figures.]

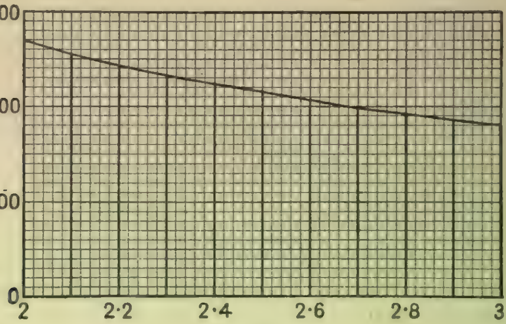


Fig. 101.

Ex. 5. If in the last example the pressure and volume were connected by a law of the form $pv^n = \kappa$ (constant), say $pv^{1\cdot 13} = \kappa$, we should get in exactly the same way

$$\text{Work done} = \int_2^3 \frac{\kappa}{v^{1\cdot 13}} \, . \, dv,$$

where $$\kappa = 2160 \times \left(\frac{1}{4}\right)^{1\cdot 13} = \frac{2160}{4^{1\cdot 13}} = \frac{540}{4^{\cdot 13}}.$$

Now $$\int \frac{1}{v^{1\cdot 13}} \, dv = \int v^{-1\cdot 13} \, dv = -\frac{1}{\cdot 13} v^{-\cdot 13}$$

$$= -\frac{1}{\cdot 13 v^{\cdot 13}}.$$

$$\therefore \text{ Work done} = \left[-\frac{\kappa}{\cdot 13 v^{\cdot 13}} \right]_2^3$$

$$= \frac{\kappa}{\cdot 13} \left\{ \frac{1}{2^{\cdot 13}} - \frac{1}{3^{\cdot 13}} \right\}$$

$$= \frac{540}{\cdot 13} \left\{ \frac{1}{8^{\cdot 13}} - \frac{1}{12^{\cdot 13}} \right\}$$

$$= \frac{540}{\cdot 13} \{ \cdot 7631 - \cdot 7239 \}$$

$$= 163 \text{ ft.-lbs. wt.}$$

EXERCISES. L.

1. The speed of a body (v ft./sec.) at the end of t secs. from a fixed instant is given by

$$v=u+at$$

[where u and a are constants].

Shew that the distance travelled in these t seconds is

$$ut+\frac{1}{2}at^2.$$

Shew also that u is the speed at the fixed instant and that the acceleration is constant and equal to a.

2. If $v=3t+7t^2$, find the space passed over between the beginning of the 3rd and the end of the 5th second.

3. If $a=3t+7t^2$ [a ft./sec.^{2} is acceleration], find the increase of speed between the end of the 4th and the end of the 8th second.

Also, if the speed when $t=0$ is 20 ft./sec., find the speed at the end of the 4th second.

Find a formula for the speed at the end of t seconds and the distance passed over between the end of the 4th and the end of the 8th second.

4. Find the work done in the expansion of a quantity of steam from 2 cubic feet at 4000 lbs. per sq. ft. pressure to 8 cubic feet. The steam expands so as to satisfy the law $pv^{0\cdot9}=$constant.

5. The pressure (p lbs. wt. per sq. ft.) and the volume (v cubic feet) of a gas are connected by the law $pv^{1\cdot7}=$constant.

When the volume is 40 cubic feet the pressure is 100 lbs. wt. per sq. ft.

Find the work done in compressing the gas from 40 to 35 cubic feet.

6. Find the resultant thrust on a rectangular area 6 ft. by 4 ft. immersed in water with its long sides vertical and the upper side 3 feet below the surface.

7. Find the resultant thrust on a triangular plate immersed in water with its plane vertical, its vertex A in the surface, and its base BC horizontal.

BC$=4$ ft. and the perpendicular from A to BC$=3$ ft.

8. Same with BC in the surface.

9. Find the work done in stretching an elastic string from length a ft. to length b ft. given that the natural length is l ft. and the pull when it is stretched to twice its natural length is k lbs. wt.

10. A ship of 600 tons displacement goes a distance of 900 feet after steam has been shut off. Supposing that the resistance of the water varies as the square of the distance which it has to go before coming to rest and that the initial resistance is 15 lbs. wt. per ton, find the work done before the ship comes to rest.

This will be the initial Kinetic Energy of the ship. Hence find the initial speed in knots.

11. A body of mass 5 lbs. moves in a straight line under the action of a force F lbs. wt. which obeys the law

$$F = 2t + 1.$$

Given that the body starts from rest, i.e. $v = 0$ when $t = 0$, find (i) the speed at the end of 5 seconds, (ii) the distance travelled in 5 seconds.

12. A body of mass m lbs. moves in a straight line so that its acceleration (a ft./sec.^{2}) is given by $a = -\omega^2 x$, where ω is a constant and x ft. is the distance from a fixed point O in the line.

Find the work done against the force as the body moves from O to a distance r.

If the speed at distance r is zero, what is the speed at O?

13. Supposing the force of the earth's attraction at points above the surface to vary inversely as the square of the distance from the earth's centre, find the work done in moving a body of mass m lbs. from the earth's surface to a height of 4000 miles. [Take earth's radius = 4000 miles.]

14. Find the weight of a rod AB 10 feet long, cross-section 1 sq. in., whose density changes from point to point in such a way that it is proportional to the distance from one end A, given that the density at B is 1 lb. per cubic inch.

Volumes of solids of revolution.

185. If a plane area be made to rotate about any line in its plane a solid of revolution is generated. Familiar instances of such solids are the cylinder, cone and sphere.

A cylinder is generated by the rotation of a rectangle about one side.

A cone is generated by the rotation of a right-angled triangle about one of the sides containing the right angle.

A sphere is generated by the rotation of a semi-circle about the diameter.

186. To find the volume of a solid of revolution.

Ex. 1. OK is the line $y=\dfrac{x}{2}$; AH, BK are the ordinates $x=2$, $x=9$. The area AHKB (Fig. 102) is rotated through a complete revolution about OX generating a frustum of a cone (Fig. 103). Required the volume of this solid.

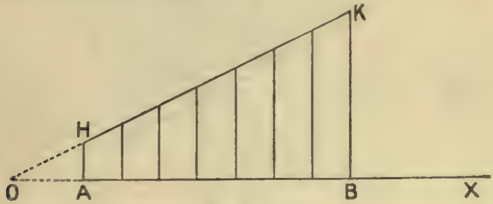


Fig. 102.

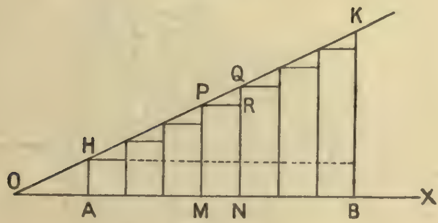


Fig. 102 *a.*

Suppose the area AHKB divided into strips by lines parallel to O*y* and the inner rectangles completed as in Fig. 102 *a*.

If the area composed of these rectangles rotate about OX we shall get a solid as in Fig. 103 *a* consisting of a number of cylinders, and just as the sum of the rectangles in Fig. 102 *a* can be made as

near as we like to the area AHKB in Fig. 102, if the strips are made thin enough, so the sum of the cylinders in Fig. 103 a can be made as near as we like to the volume required (Fig. 103).

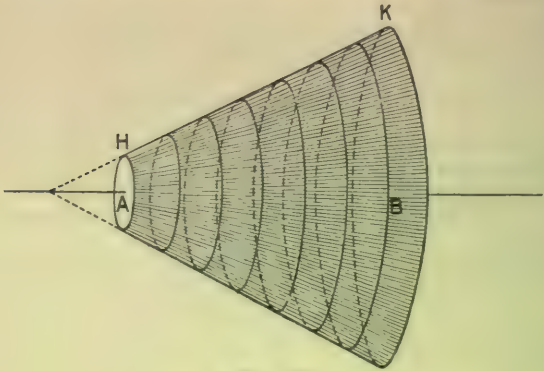


Fig. 103.

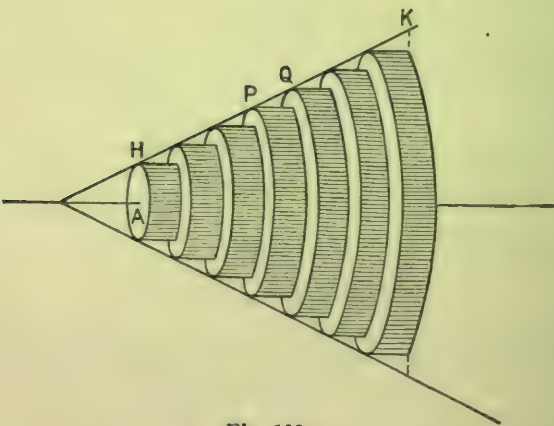


Fig. 103 a.

Now if OM $=x$ and MN $=\Delta x$, MP $=y=\frac{x}{2}$ and the volume of the typical slice in Fig. 103 a formed by the revolution of PMNR

$$=\pi \,.\, \text{MP}^2 \,.\, \text{MN}=\pi \,.\, y^2 \,.\, \Delta x=\pi \,.\, \frac{x^2}{4} \,.\, \Delta x.$$

$$\therefore \text{ Sum of slices in Fig. 103 } a=\sum_{x=2}^{x=9} \pi \frac{x^2}{4} \Delta x \text{ approximately,}$$

and actual volume of solid in Fig. 103

$$=\int_2^9 \pi \,.\, \frac{x^2}{4}\, dx=\left[\frac{\pi}{4} \,.\, \frac{x^3}{3}\right]_2^9=\frac{\pi}{12}(729-8)$$

$$=\frac{721}{12}\pi.$$

Ex. 2. To find volume of sphere, radius a.

The sphere may be considered as generated by the rotation of the area ABA′ about OX. [Fig. 104.]

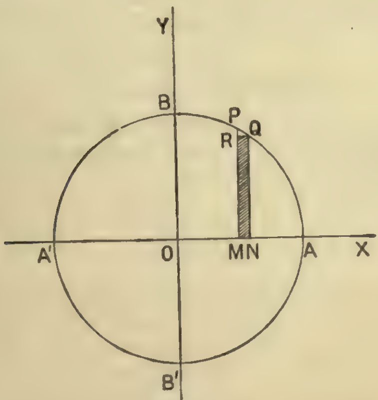


Fig. 104.

Suppose the area split into strips by lines parallel to OY. PMNQ is such a strip.

When the rotation about A′A takes place this strip will generate a slice of the sphere.

$$\text{Volume of slice} = \pi \,.\, PM^2 \,.\, MN \text{ approximately}$$
$$= \pi y^2 \Delta x \text{ approximately}$$
$$= \pi (a^2 - x^2) \Delta x \text{ approximately.}$$

(v. § 147.)

$$\therefore \text{ Volume of sphere} = \int_{-a}^{a} \pi (a^2 - x^2)\, dx$$
$$= \left[\pi\left(a^2x - \frac{x^3}{3}\right)\right]_{-a}^{a}$$
$$= \pi\left\{a^2 \times 2a - \frac{2a^3}{3}\right\} = \frac{4}{3}\pi a^3.$$

EXERCISES. LI.

1. Find the volume of a cone—radius of base 6″, height 10″.

2. Find the volume of a cone—radius of base r'', height h''.

3. In a sphere of radius 6″ find the volume of a cap of height 2″.

4. In a sphere of radius r'' find the volume of a cap of height h''.

5. In a sphere of radius 10″ find the volume of a slice contained between two parallel planes at distances 2″ and 5″ from the centre (i) on the same side, (ii) on opposite sides of the centre.

6. Find the volumes of the solids formed by the revolution of $\frac{x^2}{9} + \frac{y^2}{4} = 1$, (i) about OX, (ii) about OY.

[These solids are called respectively oblate and prolate spheroids.]

7. Find the volume of that portion of the first solid in Qu. 6 which is cut off by planes perpendicular to the axis of rotation through the points $(-1, 0)$, $(2, 0)$.

8. Find the volume of the solid formed by the rotation about OY of that portion of the parabola $y = \frac{x^2}{a}$ which lies between the origin and $y = k$. Shew that it is half the volume of the circumscribed cylinder.

This solid is called a paraboloid of revolution.

9. Trace roughly the curve $36y^2 = x(4-x)^2$.

Find the area of the loop and the volume obtained by revolving the loop about OX.

10. Find the equation of the parabolic arc with axis parallel to axis of y which passes through the points $(-2, 3)$, $(0, 5)$, $(2, 3)$.

Find the volume formed by rotating the area bounded by this arc and the extreme ordinates about OX.

11. Find the volume of the solid formed by the rotation (i) about OX, (ii) about OY of that part of $xy = 20$ which lies between $x = 2$ and $x = 5$.

12. Shew that the volume formed by the rotation about OX of that portion of $y = f(x)$ [where $f(x)$ denotes any function of x] which lies between $x = a$ and $x = b$ is $\int_a^b \pi y^2 . dx$.

Moments of inertia.

187. Definition. If a particle of mass m be placed at a distance r from a fixed axis, its moment of inertia about that axis is mr².

If a number of particles $m_1, m_2, m_3 \ldots$ be placed at distances $r_1, r_2, r_3 \ldots$ from a fixed axis, the moment of inertia of all these masses about the fixed axis is

$$m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots \text{ or shortly } \Sigma mr^2.$$

If instead of a series of disconnected particles we have a continuous body, we split it up into strips or slices as when we were finding areas or volumes, find the sum of the moments of inertia of all the pieces and the limit to which this sum tends as the pieces are indefinitely diminished.

188. *Ex.* Find the moment of inertia about a short side of a thin plate of length 3 feet and breadth 2 feet, the mass being 20 lbs.

XX′ is the axis about which the M.I. is required. [Fig. 105.]

PMNQ is a thin strip. $AP = x$ feet. $PQ = \Delta x$.

$$\therefore \text{ area PMNQ} = 2\Delta x \text{ square feet.}$$

$$\therefore \text{ mass PMNQ} = \frac{20}{6}(2\Delta x) = \frac{20}{3}\Delta x \text{ lbs.,}$$

and the distance of any point in the strip from XX′ lies between x and $x+\Delta x$.

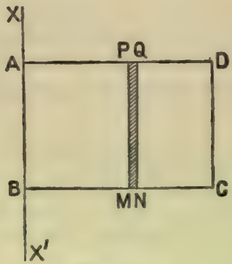


Fig. 105.

$\therefore$ The M.I. of the strip $= \dfrac{20}{3}\Delta x \,.\, x^2$ approximately.

(v. § 146.)

$\therefore$ M.I. of rectangle

$$= \int_0^3 \frac{20}{3} x^2 dx = \left[\frac{20}{9} x^3\right]_0^3 = \frac{20\times 27}{9} = 60 \text{ lb. ft.}^2$$

189. Radius of gyration. If the M.I. of a body of mass M about an axis be Mk^2, k is called the **radius of gyration** about this axis.

In this example since the mass is 20 lbs., $k^2=3$, i.e. radius of gyration $=\sqrt{3}$ feet.

190. Sometimes a more convenient method of splitting into elements than by lines parallel to an axis may be found.

e.g. Find the moment of inertia of a circular disc radius a about an axis through the centre perpendicular to its plane.

We naturally seek for a set of points at the same distance from the axis and this suggests splitting the disc into thin concentric rings.

Such a ring is shewn in figure 106.

Let $OP = x$, $PQ = \Delta x$.

Then area of ring $= 2\pi x \Delta x$ approximately, and if m be mass of unit area

$$\text{Mass of ring} = 2\pi x m \Delta x \text{ approximately,}$$

$$\text{M.I. of ring} = 2\pi x m \Delta x \times x^2 \text{ approximately,}$$

$$\therefore \text{ M.I. of disc} = \int_0^a 2\pi x^3 m\, dx$$

$$= \left[2\pi m \cdot \frac{x^4}{4}\right]_0^a = 2\pi m \cdot \frac{a^4}{4}$$

$$= \frac{m\pi a^4}{2}.$$

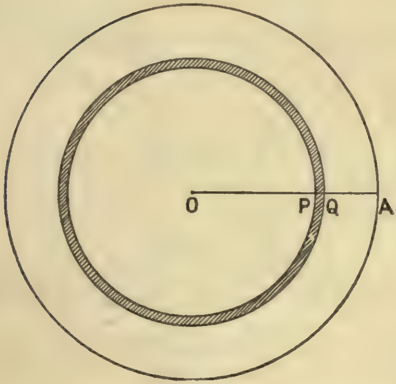


Fig. 106.

Now if M be mass of disc, $M = m\pi a^2$.

$$\therefore \text{ M.I. of disc} = \frac{Ma^2}{2}.$$

$$\text{Radius of gyration} = \sqrt{\frac{a^2}{2}} = \frac{a}{\sqrt{2}}.$$

191. Suppose we try to find the M.I. of a circular disc radius a about a diameter.

Take the diameter as the y-axis.

Here points equidistant from the axis lie on a parallel line and this suggests division into strips by lines parallel to OY. [Fig. 107.]

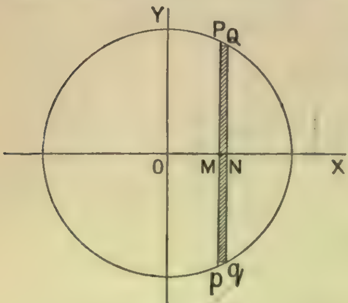


Fig. 107.

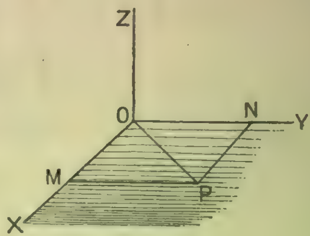


Fig. 108.

Let PpqQ be such a strip, and let $OM = x$, $MN = \Delta x$, $MP = y \left(= \sqrt{a^2 - x^2}\right)$.

$$\text{Mass of strip} = 2y\,\Delta x \times m \text{ approximately}$$

$$= 2m\sqrt{a^2 - x^2}\,.\,\Delta x.$$

$$\text{M.I. of strip} = 2m\sqrt{a^2 - x^2}\,\Delta x \times x^2 \text{ approximately},$$

and $$\text{M.I. of disc} = \int_{-a}^{a} 2mx^2\sqrt{a^2 - x^2}\,dx.$$

Here we have a function which at present we do not know how to integrate.

192. We can avoid the difficulty by making use of the following important theorem.

If OX, OY are two perpendicular axes in the plane of a lamina and OZ is a third axis perpendicular to the plane, and if the moments of inertia about OX, OY, OZ are respectively I_x, I_y, I_z, then

$$I_z = I_x + I_y.$$

Suppose a mass m situated at P in the plane XOY. [Fig. 108.]

Draw PM, PN perpendicular to OX, OY and join OP.

Then OP is perpendicular to OZ.

$$\text{M.I. of mass about OX} = m \,.\, \text{PM}^2.$$

$$\text{M.I. of mass about OY} = m \,.\, \text{PN}^2.$$

$$\text{M.I. of mass about OZ} = m \,.\, \text{PO}^2.$$

But $$\text{PO}^2 = \text{PM}^2 + \text{PN}^2.$$

$\therefore$ M.I. of mass about OZ = sum of M.I.'s about OX and OY.

This is true for any number of masses in the plane XOY whence the theorem follows.

193. To apply this to the present case.

Take an axis OZ perpendicular to the plane of the disc. We have just found that $\text{I}_z = \frac{\text{M}a^2}{2}$ (§ 190).

$$\therefore \text{I}_x + \text{I}_y = \frac{\text{M}a^2}{2}.$$

But by symmetry $\text{I}_x = \text{I}_y$.

$$\therefore \text{I}_x = \text{I}_y = \frac{\text{M}a^2}{4}.$$

[N.B. This shews that

$$\int_{-a}^{a} 2mx^2\sqrt{a^2 - x^2}\,dx = \frac{m\pi a^2 \,.\, a^2}{4},$$

or $$\int_{-a}^{a} x^2\sqrt{a^2 - x^2}\,dx = \frac{\pi a^4}{8}.\;]$$

EXERCISES. LII.

1. Find the moment of inertia of a uniform rod 10 feet long, mass 20 lbs., about an axis perpendicular to its length (*a*) through one end, (*b*) through a point 3 feet from one end.

2. Find the M.I. about AB of a rectangular plate ABCD in which AB = *a* ft., BC = *b* ft., the mass being M lbs.

3. Find the M. I. of the same plate about a line parallel to AB and DC and midway between them.

4. Find the M. I. of the same plate about a line perpendicular to its plane (i) through A, (ii) through the centre.

5. Find the M. I. of a sphere, radius a, mass M, about a diameter.

6. Find the radius of gyration about OY (i) of a lamina in the form of part of the parabola $y=\frac{x^2}{a}$ bounded by the curve and $y=k$, (ii) of the solid formed by the revolution of this area about OY.

7. Find the M. I. of a rod AB length a ft. about a line through O perpendicular to the plane OAB. O is on right bisector of AB at distance p.

8. Find the radius of gyration about the axis of (i) a hollow cone, (ii) a solid cone (radius r, height h).

9. Find the moment of inertia of a triangular lamina ABC of mass m lbs. about the side BC, given BC=2 ft. and the perpendicular from A to BC=3 ft. [First find the length of a line parallel to BC at a distance x ft. from it and terminated by the other sides.]

10. Find the radius of gyration about each side of a triangular lamina, mass M, sides 4, 5, 6 feet.

11. Find the M. I. of a solid cylinder about the axis of figure.

12. Find the M. I. of a hollow cylinder, internal and external radii R and r, about the axis of figure.

13. A grindstone, radius 1 foot, mass 100 lbs., is rotating about its axis at 100 revolutions per minute. Find its kinetic energy.

14. Find the moment of inertia about the axis of revolution of the solid formed by the revolution of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about (i) the x-axis, (ii) the y-axis.

15. Find the moment of inertia about its axis of a frustum of a cone, the radii of the ends being 3 ft. and 4 ft. and the thickness 5 feet.

Centre of gravity.

194. If we have a number of particles of masses m_1, m_2, m_3 &c. situated in a plane at points whose co-ordinates referred to some fixed axes in the plane are (x_1y_1), (x_2y_2), (x_3y_3) &c., and if $\bar{x}$ and $\bar{y}$

are the co-ordinates of the centre of gravity of the masses [Fig. 109],

then $$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \dots}{m_1 + m_2 + m_3 + \dots} \text{ or } \frac{\Sigma mx}{\Sigma m},$$

and $$\bar{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \dots}{m_1 + m_2 + m_3 + \dots} \text{ or } \frac{\Sigma my}{\Sigma m}.$$

[If we call m_1x_1 the moment of the mass m_1 about OY, these results merely express the fact:

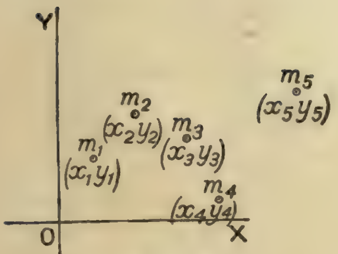


Fig. 109.

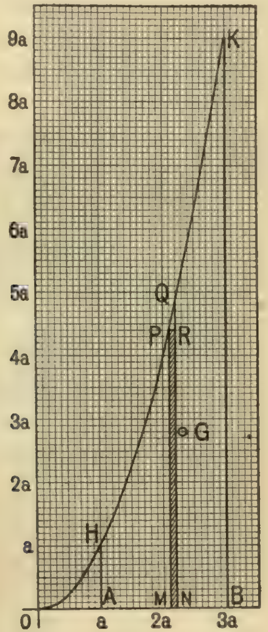


Fig. 110.

Moment about OY (or OX) of sum of masses supposed collected at centre of gravity = sum of moments about OY (or OX) of the separate masses, for this is equivalent to

$$(\Sigma m) \times \bar{x} = m_1x_1 + m_2x_2 + \dots = \Sigma mx.]$$

Now suppose we are dealing with a continuous body instead of a set of disconnected particles, we proceed as when we required the area or volume, that is to say, suppose the body split up into suitable strips or slices.

195. The method will be best seen by examples.

Ex. 1. HK is a portion of the curve $y=\frac{x^2}{a}$; AH and BK are the ordinates $x=a$, $x=3a$. [Fig. 110.]

Find the centre of gravity of the area ABKH.

Suppose the area divided into strips like PMNQ by lines parallel to OY.

Let OM $=x$, MN $=\Delta x$, MP $=y=\frac{x^2}{a}$.

Let m be the mass of unit area.

Area of strip $=y\Delta x$ approximately. (v. § 146.)

Mass of strip $=my\Delta x$ approximately.

Moment of mass about OY $=my\Delta x\times x$ approximately.

Moment of mass about OX $=my\Delta x\times\frac{y}{2}$ approximately.

Sum of masses of strips $=\sum\limits_{x=a}^{x=3a} my\Delta x$ approximately.

Sum of moments about OY $=\sum\limits_{x=a}^{x=3a} mxy\Delta x$ approximately.

Sum of moments about OX $=\sum\limits_{x=a}^{x=3a} m\frac{y^2}{2}\Delta x$ approximately.

$$\therefore \text{ Mass of ABKH} = \int_a^{3a} my\,dx = \int_a^{3a} m\frac{x^2}{a}\,dx = \frac{26}{3}ma^2.$$

$$\text{Moment of mass about OY} = \int_a^{3a} mxy\,dx = \int_a^{3a}\frac{mx^3}{a}\,dx = 20ma^3.$$

$$\text{Moment of mass about OX} = \int_a^{3a} m\frac{y^2}{2}\,dx$$

$$= \int_a^{3a}\frac{mx^4}{2a^2}\,dx = \frac{121}{5}ma^3.$$

$$\left.\begin{aligned}\therefore\quad \bar{x}&=\frac{20ma^3}{\frac{26}{3}ma^2}=\frac{30}{13}a\\ \bar{y}&=\frac{\frac{121}{5}ma^3}{\frac{26}{3}ma^2}=\frac{363}{130}a\end{aligned}\right\}.$$

Ex. 2. HK is a portion of the curve $y=\frac{x^2}{a}$; CH and DK are the lines $y=a$, $y=9a$. [Fig. 111.]

Find the centre of gravity of the solid formed by a complete revolution of the area CHKD about OY.

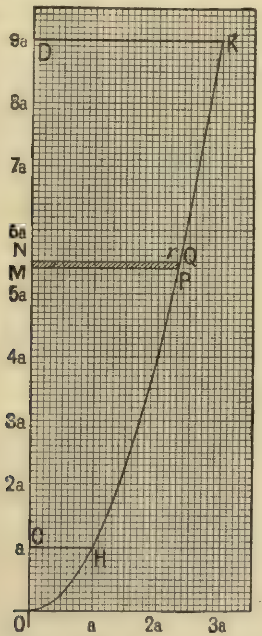


Fig. 111.

Divide the area into strips like PMNQ by lines parallel to OX. Let OM $= y$, MN $= \Delta y$ and MP $= x\,(= \sqrt{ay})$.

When the area is rotated about OY this strip will generate a slice of the solid which will be approximately a cylinder, radius of base MP, height MN.

$$\text{Volume of slice} = \pi\,.\,\text{MP}^2\,.\,\text{MN} = \pi x^2 \Delta y \text{ approximately}$$
$$= \pi ay\,.\,\Delta y \text{ approximately.}$$

$\therefore$ If m be mass of unit volume,

$$\text{Mass of slice} = m\pi ay\,.\,\Delta y \text{ approximately.}$$

$$\text{Moment of slice about OX} = m\pi ay\Delta y \times y \text{ approximately}$$
$$= m\pi ay^2 \Delta y \text{ approximately.}$$

$$\therefore \text{Mass of solid} = \int_a^{9a} m\pi ay\,dy = \left[\frac{m\pi ay^2}{2}\right]_a^{9a} = \frac{m\pi a \times 80a^2}{2}$$
$$= 40m\pi a^3.$$

$$\text{Moment about OX} = \int_a^{9a} m\pi ay^2 dy = \left[\frac{m\pi ay^3}{3}\right]_a^{9a} = \frac{m\pi a \times 728a^3}{3}$$
$$= \frac{728}{3} m\pi a^4.$$

$$\therefore \bar{y} = \frac{\frac{728}{3} m\pi a^4}{40m\pi a^3} = \frac{91}{15} a,$$

and by symmetry $\bar{x} = 0$.

$\therefore$ Co-ordinates of centre of gravity are $\left(0, \frac{91}{15} a\right)$.

196. Suppose we try to find the centre of gravity of the area of a quadrant of a circle, say OAB, radius a. [Fig. 112.]

$$\text{Area of strip PMNQ} = y\Delta x \text{ or } \sqrt{a^2 - x^2}\Delta x \text{ approximately.}$$

$$\text{Mass of strip} = m\sqrt{a^2 - x^2}\,\Delta x \text{ approximately.}$$

$$\text{Moment of strip about OY} = m\sqrt{a^2 - x^2}\,\Delta x \times x \text{ approximately.}$$

$$\therefore \text{ Mass of OAB} = \int_0^a m\sqrt{a^2 - x^2}\,dx.$$

$$\text{Moment about OY} = \int_0^a mx\sqrt{a^2 - x^2}\,dx.$$

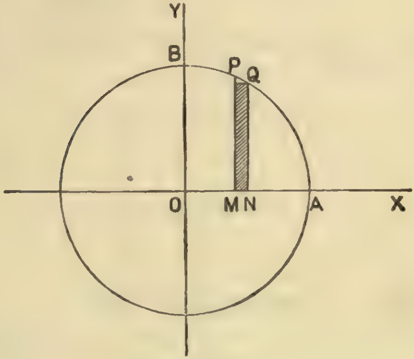


Fig. 112.

Here we have apparently two integrals neither of which we can evaluate.

The value of the first is however easily obtained, for it is $\frac{1}{4}$ of mass of whole disc $= \frac{1}{4} m\pi a^2$.

$\left[\text{N.B. This shews that } \int_0^a \sqrt{a^2 - x^2}\,dx = \frac{\pi a}{4}.\right]$

To avoid the difficulty of evaluating the second integral consider the moment of the strip about OX.

It is $\quad my\Delta x \times \frac{y}{2}$ approximately

$$= \frac{m}{2} y^2 \Delta x \text{ approximately}$$

$$= \frac{m}{2}(a^2 - x^2)\Delta x \text{ approximately.}$$

$$\therefore \text{ Moment of whole area about OX} = \int_0^a \frac{m}{2}(a^2 - x^2)\,dx$$

$$= \left[\frac{m}{2}\left(a^2x - \frac{x^3}{3}\right)\right]_0^a$$

$$= \frac{m}{2}\left\{a^3 - \frac{a^3}{3}\right\} = \frac{ma^3}{3},$$

and it is obvious from symmetry that the moment about OY is the same. [This tells us that

$$\int_0^a x\sqrt{a^2 - x^2}\,dx = \frac{1}{3}a^3.]$$

$$\therefore \bar{x} = \bar{y} = \frac{\dfrac{ma^3}{3}}{\dfrac{m\pi a^2}{4}} = \frac{4a}{3\pi}.$$

EXERCISES. LIII.

1. OP is the line $y = kx$. P is the point on it whose abscissa is a.

Find by integration the co-ordinates of the C. G. of the triangle OMP where MP is the ordinate of P.

2. Find the distance from the base BC of the C. G. of a triangle ABC whose altitude is h.

3. Find the C. G. of that portion of the parabola $y = \dfrac{x^2}{a}$ cut off by the line $y = k$.

4. Find the C. G. of a cone. [Take your origin at the vertex and suppose the cone generated by the revolution of part of the line $y = kx$ about the axis of x.]

5. Find the C.G. of the area bounded by the x-axis, the ordinates $x = 3$, $x = 10$ and the curve $y = 3x^2 - 2x - 1$.

6. Find the C. G. of the trapezium whose corners are $(a, 0)$, $(-a, 0)$, (p, q), $(-p, q)$.

7. Find the C. G. of a semicircle, radius a.

8. Find the C. G. of a hemisphere, radius a.

9. The lengths of the parallel sides of a trapezium are a, b and the distance between them is h.

Shew that the C. G. lies on the line joining the mid-points of the parallel sides and that its distance from the side a is $\dfrac{h}{3}\cdot\dfrac{a+2b}{a+b}$. [First find the length of a line parallel to the side a distance x from it.] What does your result give you if $a = 0$?

10. Shew that the co-ordinates of the C. G. of a uniform lamina bounded by $y=0$, $x=a$, $x=b$ and the curve $y=f(x)$ are given by

$$\bar{x}=\frac{\int_a^b xy\,.\,dx}{\int_a^b y\,dx},\quad \bar{y}=\frac{\int_a^b \frac{1}{2}y^2dx}{\int_a^b y\,dx}.$$

If the lamina is uniform this result is independent of the density and the point is sometimes called the *Centre of area* of the figure bounded by $y=0$, $x=a$, $x=b$ and the curve $y=f(x)$.

11. A lamina, area A sq. ft., is totally immersed in water with its plane vertical and its C. G. at a depth h ft. below the surface. Shew that the resultant thrust on it $=w\mathrm{A}h$ lbs. wt. where w lbs. is the weight of a cubic foot of water.

197. (i) The moment of inertia of a lamina about any axis in its plane is equal to the moment of inertia about a parallel axis through the centre of gravity + the moment of inertia about the first axis of the whole mass supposed concentrated at the centre of gravity.

Let particles of masses m_1, m_2 &c. lie in the plane of the paper at points P_1, P_2, &c., and let G be their C. G.

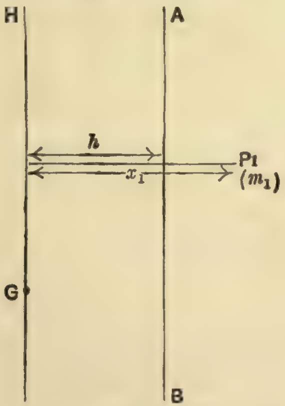


Fig. 113.

Suppose we want the M. I. of these masses about AB and let HG be a parallel axis through G.

Let x_1, x_2, &c. be the distances of the masses from HG, and let h be the distance between the two axes.

M.I. of mass m_1 about AB $= m_1(x_1 - h)^2 = m_1x_1^2 - 2m_1x_1h + m_1h^2$, and similarly for each of the other masses.

$\therefore$ M.I. of all the masses about AB $= \Sigma(mx^2) - 2h(\Sigma mx) + h^2\Sigma m$,

and M.I. " " " GH $= \Sigma(mx^2)$.

Also since G is the C.G. of the masses $\dfrac{\Sigma(mx)}{\Sigma(m)} = 0$,

$$\therefore \ \Sigma(mx) = 0.$$

$\therefore$ M.I. about AB $=$ M.I. about GH $+ h^2.(\Sigma m)$

$=$ M.I. about GH $+$ M.I. about AB of the whole mass (Σm) supposed placed at G.

Hence the theorem.

(ii) Now let O be any point in the plane of the lamina, G its C.G.; OX, OY any two axes at right angles through O; GX′, GY′ two parallel axes through G.

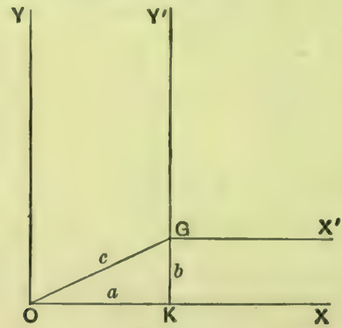


Fig. 114.

Then, if I_x be the M.I. about OX, I_x' about GX′, &c.,

we have $$I_x = I_x' + Mb^2,$$

$$I_y = I_y' + Ma^2.$$

$$\therefore \ I_x + I_y = I_x' + I_y' + Mc^2,$$

i.e. $$I_z = I_z' + Mc^2,$$

where I_z and I_z' are M.I. about axes through O and G perpendicular to the plane of the lamina.

(iii) Now let AB be parallel to the plane of the lamina, GH a parallel axis through G.

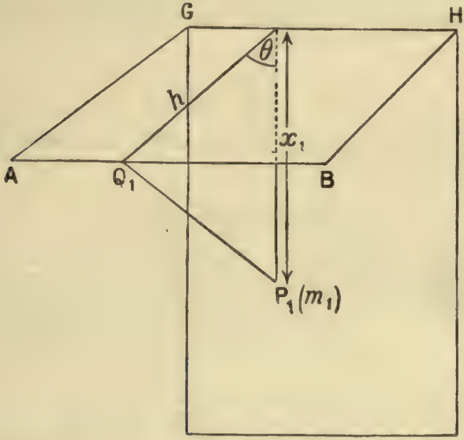


Fig. 115.

Then $P_1Q_1^2 = x_1^2 + h^2 - 2hx_1 \cos\theta$, where θ is the angle between the plane ABHG and the plane of the lamina.

$$\therefore\ \Sigma(mPQ^2) = \Sigma(mx^2) + h^2\Sigma(m) - 2h\cos\theta\,\Sigma(mx),$$

and as before $\Sigma(mx) = 0$.

So that M.I. of masses about AB = M.I. about GH + M.I. about AB of $\Sigma(m)$ placed at G.

198. Hence the moment of inertia of a plane lamina about any axis in its plane, parallel to its plane, or perpendicular to its plane = the moment of inertia about a parallel axis through the centre of gravity + the moment of inertia about the original axis of the whole mass supposed concentrated at the centre of gravity.

Ex. The M.I. of a circular disc, mass M, radius a, about an axis through the centre O, perpendicular to its plane, is

$$\frac{Ma^2}{2}.$$

If we want the M.I. about an axis through A a point on the circumference perpendicular to the plane, the theorem tells us that it is

$$\frac{Ma^2}{2} + Ma^2$$

(Ma^2 being the M.I. about A of the whole mass supposed concentrated at the centre of gravity O).

EXERCISES. LIV.

1. Find the radius of gyration of a circular lamina about a tangent.

2. Find the M.I. of a circular lamina, radius a, about a line perpendicular to its plane and meeting it in a point distance c from the centre.

3. Find the M.I. of a circular lamina, radius a, about a line in its plane distance c from the centre.

4. Find the M.I. of a thin circular hoop, radius a, about an axis perpendicular to its plane through a point in the rim.

5. Find the M.I. of a circular lamina about a line parallel to its plane at a distance c from the centre.

6. Find the M.I. of a solid cylinder about a line through the centre perpendicular to the axis of figure.

7. Find the radius of gyration of a solid cone about a line through the vertex perpendicular to the axis.

Centre of pressure.

199. In Ex. 3, § 184 we found the resultant thrust on a rectangular area immersed in water.

Suppose we wish to find through what point of the plate this resultant pressure acts. [This point is called the **centre of pressure.**]

Our approximate solution would be the finding of the line of action of the resultant of the thrusts on the strips into which the rectangle is supposed to be divided.

The principle of which we make use is that if we take moments about any line, the moment of the resultant of any number of forces = the sum of the moments of the separate forces.

Now it is obvious that the centre of pressure lies on the vertical line of symmetry. It therefore remains to find its distance from AB.

Take moments about AB.

The thrust on our typical strip PQNM $= 250x\Delta x$ lbs. wt. approximately and may be taken to act at the centre of the rectangle PQNM.

$\therefore$ Moment of this thrust about AB $= 250x\Delta x \times x$ lbs. ft. approximately and the sum of the moments of all the thrusts

$$= \int_0^6 250x^2 . dx = \left[\frac{250}{3}x^3\right]_0^6 = 250 \times 72 \text{ lbs. ft.}$$

The resultant thrust we have seen to be

$$\int_0^6 250x dx = \left[\frac{250}{2} . x^2\right]_0^6 = 250 \times 18 \text{ lbs. wt.}$$

If X ft. be the distance below AB of the centre of pressure the moment of this resultant thrust

$$= 250 \times 18 \times \mathrm{X} \text{ lbs. ft.}$$

$$\therefore 250 \times 18 \times \mathrm{X} = 250 \times 72.$$

$$\therefore \mathrm{X} = 4.$$

$\therefore$ Centre of pressure is 4 ft. below the surface.

EXERCISES. LV.

1. Find the centre of pressure of a rectangle a feet by b feet, placed in water with its plane vertical and one of the a feet sides (i) in the surface, (ii) c feet below the surface.

2. Find the centre of pressure of a triangle, base a feet, height h feet, placed in water with its plane vertical and (i) its vertex, (ii) its base in the surface, (iii) its vertex c feet and its base $(c+h)$ feet below the surface, (iv) its base c feet and its vertex $(c+h)$ feet below the surface.

3. The parallel sides of a trapezium are a, b and the distance between them is h. It is placed in water with its plane vertical; find the centre of pressure if the side a is (i) in the surface, (ii) c ft. below the surface.

Mean values.

200. We have already (§ 164) defined the mean value of $f(x)$ between $x=a$ and $x=b$ as being the height of a rectangle whose base is $(b-a)$ and whose area is equal to that bounded by the curve $y=f(x)$, the x-axis and the ordinates $x=a$ and $x=b$. This mean value is

$$\frac{\int_a^b f(x) \,.\, dx}{b-a},$$

and we might take this as the definition of the mean value.

It will be instructive to obtain this result by a method which does not involve the idea of area.

In ordinary arithmetic the mean or average of a set of quantities is obtained by dividing the sum of the quantities by their number [i.e. it is the Arithmetic mean of the quantities].

Thus if 5 men have heights 5 ft. 8 ins., 5 ft. 6 ins., 6 ft., 6 ft. 2 ins. and 5 ft. 2 ins. the average height is

$$\frac{5'\ 8'' + 5'\ 6'' + 6' + 6'\ 2'' + 5'\ 2''}{5} = \frac{28'\ 6''}{5} = 5'\ 8\tfrac{2}{5}''.$$

The combined height of 5 men each of height 5 ft. $8\frac{2}{5}$ ins. is the same as that of the given 5 men.

Now suppose we have a function of x, $f(x)$, and suppose x to pass from the value a to the value b by n equal steps each $=h$ so that $nh=b-a$.

Successive values of x are

$$a,\ a+h,\ a+2h,\ \ldots\ (b-h),\ b,$$

and the corresponding values of $f(x)$ are

$$f(a),\ f(a+h),\ f(a+2h),\ \ldots f(b-h),\ f(b),$$

$(n+1)$ in number.

The Arithmetic mean of these values is

$$\frac{f(a)+f(a+h)+f(a+2h)+\ldots+f(b-h)+f(b)}{n+1},$$

or
$$\frac{hf(a)+hf(a+h)+\dots+hf(b-h)+hf(b)}{(b-a)+h},$$
since $nh=b-a$.

We define the mean value of $f(x)$ between a and b as the limit of this when $h \to 0$, i.e. when the number of steps from a to b is indefinitely increased. We have seen that the numerator can be written
$$\sum_{x=a}^{x=b} f(x)\,.\,\Delta x,$$
where Δx takes the place of h and the limit of this when $\Delta x \to 0$ is
$$\int_a^b f(x)\,dx.$$

Also the limit of the denominator is $b-a$.

$\therefore$ The mean value of $f(x)$ between a and b is
$$\frac{\int_a^b f(x)\,.\,dx}{b-a}.$$

201. In questions of mean value it is important to make clear what the independent variable is.

The following example will illustrate what is meant.

A stone falls to the ground from the top of a tower 400 feet high. Find its mean speed. [$g=32$.]

We may think of the speed as a function of either (i) the time, or (ii) the distance fallen.

If we take (i) we suppose the whole time of fall divided into n equal parts, the speeds at the ends of these equal intervals of time calculated, and the Arithmetic mean of them obtained. Our mean speed is the limiting value of this Arithmetic mean when n is indefinitely increased.

It is easily found that 5 secs. is the time of falling and that $v=32t$ is the formula which expresses v as a function of t.

∴ The mean speed on the first assumption is

$$\frac{\int_0^5 vdt}{5} = \frac{\int_0^5 32t \, . \, dt}{5} = \frac{\left[16t^2\right]_0^5}{5} = \mathbf{80} \text{ ft./sec.}$$

If we take (ii) we suppose the whole distance, 400 feet, divided into n equal parts, the speeds at the points of division calculated and the Arithmetic mean obtained.

Our mean speed is the limiting value of this Arithmetic mean when n is indefinitely increased.

The formula connecting v and s is $v^2 = 64s$.

∴ The mean speed on the second assumption is

$$\frac{\int_0^{400} \sqrt{64s} \, . \, ds}{400} = \frac{1}{50}\left[\frac{2}{3} \, . \, s^{\frac{3}{2}}\right]_0^{400} = \frac{1}{50} \cdot \frac{2}{3} \cdot 20^3 = \mathbf{106\tfrac{2}{3}} \text{ ft./sec.}$$

202. *Ex.* 1. Find the average speed between the end of the 2nd and the end of the 5th second of a body whose speed changes according to the law $v = 6t + 17t^2$.

i.e. find the mean value of $6t + 17t^2$ between $t = 2$ and $t = 5$.

This is $$\frac{\int_2^5 (6t + 17t^2) \, dt}{5 - 2} = \frac{726}{3} = \mathbf{242} \text{ ft./sec.}$$

Since $\int_2^5 (6t + 17t^2) \, dt$ or 726 ft. is the distance travelled in the 3 seconds between the end of the 2nd and the end of the 5th second, this speed of 242 ft./sec. is the uniform speed with which the body would have to move to cover the same distance in the same time (v. definition of average speed in Chap. I.).

Ex. 2. The pressure and volume of a gas are connected by the law $pv = 540$, p being in lbs. wt./sq. ft. and v in cub. ft.

Find the mean pressure as the gas expands from 2 to 3 cub. ft.

i.e. find the mean value of $\frac{540}{v}$ between $v = 2$ and $v = 3$.

This is $$\frac{\int_2^3 \frac{540}{v} \,.\, dv}{3-2} = 219 \text{ lbs. wt./sq. ft.} \quad [\text{v. Ex. 4, § 184.}]$$

This is the uniform pressure which would have to act on the piston to do the same amount of work that is actually done.

EXERCISES. LVI.

1. Find the mean sectional area of a sphere supposed cut by a series of equidistant parallel planes.

Interpret the result geometrically.

2. Find the mean sectional area of a cone supposed cut by a series of planes parallel to the base. [Radius of base r, height h.]

3. If $v=u+at$ where u and a are constants, shew that the average speed between the end of t_1 seconds and the end of t_2 seconds is equal to half the sum of the speeds at these instants.

4. If $v=u+at+bt^2$ where u, a, b are constants, find the average speed between the end of t_1 seconds and the end of t_2 seconds.

Also find half the sum of the speeds at these instants.

5. The pressure and volume of a gas are connected by the law

$$pv^{1\cdot 2}=\text{constant}.$$

If the pressure is 400 lbs. per sq. ft. when the volume is 3 cub. ft., find the mean pressure as the gas expands from 2 to 4 cub. ft.

6. Find the mean gradient (i) with respect to x, (ii) with respect to y, of $y=x^2$, between the points (2, 4), (5, 25).

7. A stone falls freely from rest; shew that the mean speed with respect to the time is half the final speed, but the mean speed with respect to the distance is two-thirds of the final speed.

8. A stone falls freely from rest. Shew that the mean kinetic energy with respect to the time is two-thirds of the mean kinetic energy with respect to the distance fallen.

MISCELLANEOUS EXAMPLES ON CHAPTERS VII. AND VIII.

A.

1. Draw roughly the curve $y=6-4x-x^2$ cutting the x-axis in A, B and the y-axis in C.

Find the areas AOC, BOC.

2. The parabola $y^2=6x$ forms a paraboloid of revolution by revolving about the x-axis. Find the volume of a segment of this paraboloid of length 10 units, and shew that it is half the volume of the circumscribing cylinder.

3. Find the mean value of $10x^2-x^3$ between $x=3$ and $x=5$. Explain your result graphically.

4. A point starts with initial velocity u feet per second, and moves along a straight path with a constant acceleration a feet per second per second. Prove by integration that its displacement, s, at any time t secs. after the start is given by

$$s=\left(ut+\frac{1}{2}at^2\right) \text{ feet.}$$

5. ABC is a triangle having a right angle at C; BC is horizontal, and A, which is vertically above BC, is in the surface of a liquid; find the centre of pressure.

If the triangle were turned about BC until A were vertically below C, find where the centre of pressure would now be.

B.

1. Write down the values of

$$\int (2x^2-3x+1)\,dx\,; \quad \int x\sqrt{x}\,dx\,; \quad \int \frac{dx}{x^2}.$$

2. Sketch the curve $y=x(5-x)^2$ between $x=-1$ and $x=6$.

Find the area bounded by the x-axis and that portion of the curve which lies between $x=0$ and $x=5$.

Find also the maximum ordinate and the area of the circumscribed rectangle.

3. The equation of an ellipse is $x^2+4y^2=4$. Shew that the volume generated by the rotation of this ellipse about the x-axis is half the volume generated by its rotation about the y-axis.

4. Find the abscissae of the point in which the curve $y=x^3+x^2-20x$ cuts the x-axis. Calculate the maximum and minimum ordinates and the gradient at each of the points where the curve cuts the x-axis.

Sketch the curve roughly between the extreme points in which it cuts the x-axis. Find the areas of the two portions in the 2nd and 4th quadrants.

5. Find the Moment of Inertia of a circular disc, radius a, about a line parallel to its plane at a distance c from its centre.

Hence find the Moment of Inertia of a solid cone about a line through its C.G. perpendicular to the axis.

C.

1. Consider only the part of the curve $y=4x-x^3$, for which neither x nor y is negative.

Make a rough sketch of this part.

Find the equations to the tangents at the points where it cuts the axis of x.

Find the length of its greatest ordinate.

Find the area between the curve and the axis of x.

2. A solid is formed by the revolution about the x-axis of the curve

$$y^2=a+bx+cx^2+dx^3.$$

Find volume of solid between $x=-h$ and $x=h$ and prove that this volume is given by Simpson's rule

$$\frac{h}{3}(A_1+4A_2+A_3),$$

where A_1 and A_3 are the areas of the end sections, A_2 of the middle section (corresponding to $x=0$).

3. Find the moment of inertia of a uniform thin triangular lamina ABC about the side BC.

4. A quantity of steam expands so as to follow the law $pv^{\cdot 8}=$constant. Find the work done in expanding from 1 to 4 cub. ft., given that when the volume is 2 cub. ft. the pressure is 50 lbs. per sq. ft.

Also find the mean pressure during this expansion.

5. A square side a is placed in water with a corner in the surface and the diagonal through that corner vertical. Find the depth of the centre of pressure.

D.

1. Find
$$\int_0^2 (3x^2+5x+6)\,dx, \quad \int_0^4 \frac{dx}{\sqrt{x}},$$
$$\int_1^4 \left(\frac{3}{x^{\frac{3}{2}}}+5x^{\frac{1}{2}}\right) dx.$$

2. Draw roughly $y=(x+2)(x-1)(x-4)$. If the points in which this cuts the x-axis be, in order from the left, A, B, C, find the areas bounded by the x-axis and (i) that portion of the curve between A and B, (ii) that portion between B and C.

3. Two parallel planes cut a sphere in circles whose diameters are 12 inches and $10\sqrt{3}$ inches respectively, the perpendicular distance between the planes being 13 inches. Find the radius of the sphere and shew that the planes cut off spherical segments whose volumes are in the ratio 112 : 625.

4. An isosceles triangle 10 metres in altitude is immersed in water with its base in the surface and vertex pointing downwards. Find the depth of its centre of pressure supposing the pressure at the surface to be that due to a column of water 10 m. high.

5. Find the moment of inertia about its axis of a grindstone 120 cm. in diameter, 20 cm. thick, whose density is 2·14 gms. per c.c.

Also find its kinetic energy at 15 revolutions per minute.

E.

1. Shew that
$$\int_0^3 (x^2-4x+3)\,dx=0.$$
Draw a figure to explain the result.

2. A rectangle 5 ft. by 3 ft. revolves about a straight line parallel to the longer side and 3 ft. from the nearest one. Find the volume generated and the radius of gyration of the solid so formed about the axis of revolution.

3. Indicate in a diagram the curves
$$yx^{1\cdot 2}=2 \text{ and } yx^{1\cdot 5}=2.$$
At what angle do these curves intersect each other? Find the area of the closed figure bounded by these curves and the ordinate $x=2$.

Find the co-ordinates of the centre of gravity of this area.

4. Assuming that above the surface the force of the earth's attraction varies inversely as the square of the distance from the centre and below the surface directly as the distance, find the work done in moving a mass of m lbs. from a feet below the surface to the surface, calling the earth's radius R feet; also the work done in moving the mass from the surface to a feet above.

Find the ratio of the work done in moving a mass from 1000 miles below the surface to the surface, to that done in moving it from the surface to 1000 miles above, taking the earth's radius as 4000 miles.

5. A zone of a sphere has thickness t. A cylinder has the same thickness and the area of its base equal to the Arithmetic mean of the areas of the ends of the zone. Shew that volume of the zone = the volume of cylinder + the volume of sphere, diameter t.

F.

1. Find the values of

$$\text{(i)} \int 3x^{\frac{3}{2}} . dx, \quad \text{(ii)} \int_1^3 \left(4x^2 + \frac{4}{x^2}\right) dx.$$

2. A curve has to be drawn through the points (0, 0), (1, 4), (2, 1), (3, 5). Assuming the equation to be of the form $y = ax^3 + bx^2 + cx + d$ determine a, b, c, d and find by integration the area of the curve between the extreme ordinates and the axis of x.

The area comes out to be that of a rectangle on the same base with height $2\frac{1}{2}$. Draw a figure shewing the curve and this rectangle, and shew that the result can be obtained from symmetry.

3. There is a curve $y = ax^n$.

If $y = 2{\cdot}34$ when $x = 2$ and $y = 20{\cdot}61$ when $x = 5$ find a and n.

Let the curve rotate about the x-axis forming a surface of revolution. Find the volume between the sections at $x = 2$ and $x = 5$.

4. AB is a rod l ft. long and O a point in the rod a ft. from A.

Find M.I. of the rod about an axis through O perpendicular to rod.

Find O so that this M.I. is a minimum.

5. A zone of a sphere has thickness t. A cylinder has the same thickness and the diameter of its base equal to the diameter of the section midway between the ends of the zone. Shew that the volume of the zone = the volume of the cylinder − the volume of a hemisphere, diameter t.

G.

1. Trace roughly $y=x-x^3$ from $x=0$ to $x=1$.

Find the area between it and the x-axis.

If your x-unit is 5″ and your y-unit 10″ what is the actual area in square inches?

What is the mean height of this portion of the curve?

Find also the area bounded by the curve, the x-axis and the maximum ordinate between $x=0$ and $x=1$.

2. A hemisphere 1 foot in radius has to be divided into two equal parts by a plane parallel to the base. Prove that if the distance of the plane from the centre is x feet, then x is a root of the equation

$$x^3-3x+1=0.$$

3. Prove that the area contained between the parabola $y^2=4ax$, the axis of y and the line through the point (h, k) on the parabola parallel to the x-axis is $\frac{1}{3}hk$.

Prove that the radius of gyration about the x-axis of a lamina shaped like this area is $k\sqrt{\frac{3}{5}}$ and that the co-ordinates of the C.G. are $\left(\frac{3}{10}h, \frac{3}{4}k\right)$.

4. A torque T lb. ft. acts on a shaft and is proportional to the angle turned through ($T=k\theta$).

Shew that the work done in turning the shaft through an angle a radians is $\frac{1}{2}T_1a$ ft. lbs. where T_1 is the value of the torque when the shaft has turned through an angle a.

5. A cigar-shaped strut, 40 ft. long, is of diameter $\left(10-\frac{x^2}{100}\right)$ inches at a section x feet from the middle. Find its total volume.

H.

1. Trace the curve $5y=(x+3)(x-1)(x-3)$ from $x=-4$ to $x=4$.

If it cuts the x-axis in A, B, C reading from the left and the y-axis in D, find the areas of the figures bounded by the curve and (i) OA, OD, (ii) OB, OD, (iii) BC.

Also find the mean value of y between $x=-4$ and $x=4$.

2. Find

(i) The area comprised between the x-axis and that portion of $y=6x^2-6x-12$ which lies below it.

(ii) The C.G. of the area.

(iii) The volume generated by the revolution of this area about OX.

(iv) The C.G. of this volume.

3. Find the equation of a parabola, whose axis is along the axis of y, and which passes through the three points $(-3, 2)$, $(0, 2\frac{1}{2})$, $(3, 2)$.

Use your result to find the volume of a cask, height 6 feet, equal circular ends of radii 2 feet, radius of the middle section $2\frac{1}{2}$ feet, and such that the curve of a vertical section is a parabola.

4. Find the radius of gyration of a uniform solid hemisphere about a diameter of the base.

5. A body mass 10 lbs. moves in a straight line in such a way that its acceleration at any instant varies as its distance from a fixed point O in the line and is directed towards O.

At a distance of 1 foot from O the acceleration is 64 ft./sec.^{2} Find the work done in bringing the body to O from a distance of 6 feet.

I.

1. Shew that the value of

$$\int_{-a}^{a} (5+mx)\,dx$$

is the same for all values of m. Interpret this result geometrically.

Do the same for

$$\int_{-a}^{a} (5+mx+2x^2)\,dx.$$

2. The bounding radii of a sector of a circle of radius a include an acute angle θ.

If this sector revolves about one of its bounding radii, shew that the volume of the spherical sector thus generated is

$$\frac{2}{3}\pi a^3 (1-\cos\theta).$$

3. The section of a girder is of the form shewn in the accompanying figure (Fig. 116). It is 3″ high, the greatest breadth is 2″ and the breadth of the straight part in the middle which is 1″ high is 1″.

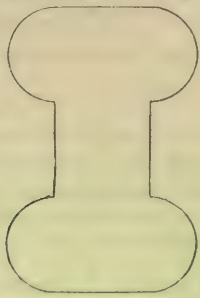


Fig. 116.

The section may thus be regarded as made up of a rectangle 3″ × 1″ with 4 semicircles 1″ in diameter.

Find the radius of gyration of the section about the axis AB through the centre perpendicular to the greatest length.

4. Shew that $$\int_{-a}^{a} x\sqrt{a^2-x^2}\,dx=0.$$

5. Find the average length of the ordinates to the parabola $y^2=8x$ erected at equidistant intervals from $x=0$ to $x=6$.

J.

1. Find the volume formed by the revolution about OX of that portion of the curve $x+y=2x^{\frac{3}{4}}$, for which x and y are both positive.

2. A hemispherical bowl radius 9″ is being filled with water at the rate of half a cubic foot per minute. Find the rate at which the depth is increasing when the water is 6 inches deep.

3. A solid iron cylinder, diameter 6 inches, length 2 feet, rotates about an axis along a diameter of one end. If it performs 100 revolutions per minute, find its K.E., given that a cubic inch of iron weighs ·28 lbs.

4. Find the area enclosed between $y^2=x^3$ and $x^2=y^3$, and the volume generated by the revolution of this area about the x-axis.

5. If the radius of the earth is r feet and a body of mass m lbs. fall from a height a feet to the earth, find the work done on it by the earth's attraction and shew that if a be very large compared with r the work done is approximately mr ft.-lbs. wt.

CHAPTER IX

DIFFERENTIATION OF TRIGONOMETRICAL RATIOS

203. We have so far confined ourselves to problems depending on the differentiation and integration of x^n.

We shall now investigate rules for dealing with other functions.

Differential coefficient of sin x.

204. Let $y = \sin x$, where x is the number of radians in the angle.

Then with the usual notation

$$y + \Delta y = \sin(x + \Delta x).$$

$$\therefore \Delta y = \sin(x + \Delta x) - \sin x$$

$$= 2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}.$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x}$$

$$= \cos\left(x + \frac{\Delta x}{2}\right) \cdot \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}.$$

Now, as $\Delta x \to 0$, $\cos\left(x + \frac{\Delta x}{2}\right) \to \cos x$ and $\dfrac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \to 1.$

(v. § 73.)

$\therefore$ The limit of $\frac{\Delta y}{\Delta x}$ when $\Delta x \to 0$ is $\cos x$.

i.e. $$\frac{dy}{dx} = \cos x.$$

Illustrations of this result.

205. (1)

Angle in degrees	x =No. of radians in $\angle$	y =sin x			
40°	·6981317	·6427876			
	$x+\Delta x$	$y+\Delta y$	Δx	Δy	$\frac{\Delta y}{\Delta x}$
41°	·7155850	·6560590	·0174533	·0132714	·760
40° 30′	·7068583	·6494480	·0087266	·0066604	·763
40° 10′	·7010406	·6450132	·0029089	·0022256	·7651
40° 5′	·6995861	·6439011	·0014544	·0011135	·7656
40° 1′	·6984226	·6430104	·0002909	·0002228	·7659

And cos 40° = ·7660.

Thus we see that as the increase in x is made smaller, $\dfrac{\text{increase in } \sin x}{\text{increase in } x}$ approaches the value $\cos x$. What we proved in § 204 is that it can be brought as near to $\cos x$ as we like if the increase in x is made small enough.

206. (2) Draw the graph of $y = \sin x$ (where x is C.M.). At a few points, say $x = 0$, $x = \cdot 7$, $x = 1$, $x = 1 \cdot 3$ draw lines whose gradient is $\cos x$ and verify that they are tangents to the graph. [Fig. 117.]

x	Angle in deg. &c.	$y = \sin x$	$\frac{dy}{dx} = \cos x$
0	0	0	1
·1	5° 44′	·100	
·2	11° 28′	·199	
·3	17° 11′	·295	
·4	22° 55′	·389	
·5	28° 39′	·479	
·6	34° 23′	·565	
·7	40° 06′	·644	·765
·8	45° 50′	·717	
·9	51° 34′	·783	
1·0	57° 18′	·842	·540
1·1	63° 02′	·891	
1·2	68° 45′	·932	
1·3	74° 29′	·964	·268
1·4	80° 13′	·985	

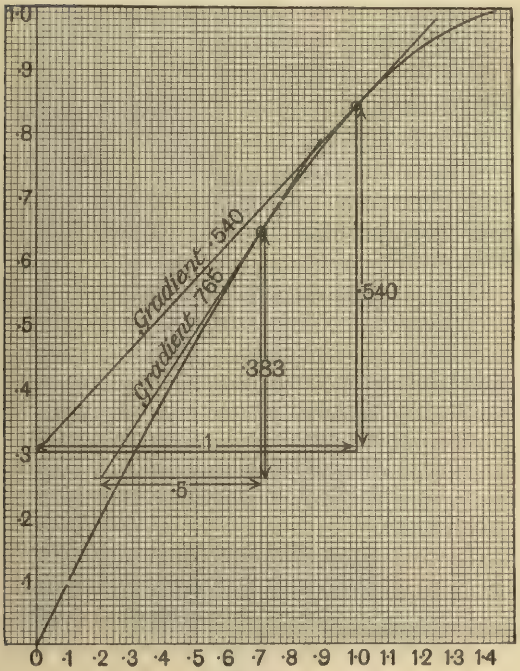


Fig. 117.

Geometrical proof.

207. P, Q are points on a circle radius r such that

$$AOP = x \text{ radians,}$$

$$POQ = \Delta x \text{ radians.} \quad \text{[Fig. 118.]}$$

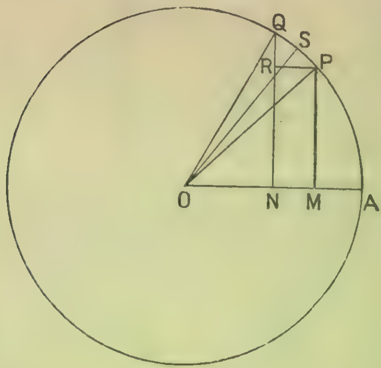


Fig. 118.

OS bisects $\angle$ POQ.

$$y = \sin x = \frac{MP}{r},$$

$$y + \Delta y = \sin \overline{x + \Delta x} = \frac{NQ}{r}.$$

$$\therefore \; \Delta y = \frac{RQ}{r},$$

and $$\Delta x = \frac{\overset{\frown}{PQ}}{r}.$$

$$\therefore \; \frac{\Delta y}{\Delta x} = \frac{RQ}{\overset{\frown}{PQ}} = \frac{RQ}{PQ} \cdot \frac{PQ}{\overset{\frown}{PQ}}.$$

Now $$\angle PQR = \angle AOS = x + \frac{\Delta x}{2}.$$

[The arms of $\angle$ PQR are perpendicular to those of $\angle$ AOS.]

$$\therefore \frac{\Delta y}{\Delta x} = \cos\left(x + \frac{\Delta x}{2}\right) \cdot \frac{\text{PQ}}{\overset{\frown}{\text{PQ}}}.$$

$$\text{As } \Delta x \to 0, \cos\left(x + \frac{\Delta x}{2}\right) \to \cos x \text{ and } \frac{\text{PQ}}{\overset{\frown}{\text{PQ}}} \to 1.$$

$$\therefore \frac{dy}{dx} = \cos x.$$

It is very important to remember that x is the number of **radians** in the angle.

208. *Ex.* 1. To find the gradient of the curve $y = \sin x$ at the point where $x = \cdot 5$. We have

$$\frac{dy}{dx} = \cos x.$$

$\therefore$ Gradient where $x = \cdot 5$ is $\cos \cdot 5$, i.e. $\cos$ ($\cdot 5$ radians)

$$= \cos 28° \ 39'$$

$$= \mathbf{\cdot 8776.}$$

To find the equation of the tangent at this point we have when $x = \cdot 5$

$$y = \sin \cdot 5$$

$$= \sin 28° \ 39'$$

$$= \mathbf{\cdot 4795.}$$

$\therefore$ We want the equation of the line through ($\cdot 5$, $\cdot 4795$) whose gradient is ($\cdot 8776$).

This is $$y - \cdot 4795 = \cdot 8776\,(x - \cdot 5),$$

or $$\mathbf{y - \cdot 8776x = \cdot 0407.}$$

Ex. 2. If $$y = \sin x,$$

$$\frac{dy}{dx} = \cos x.$$

$$\therefore \Delta y = \cos x \,.\, \Delta x \text{ approximately.}$$

e.g. if $x=\frac{\pi}{6}$ [C.M. of 30°], $\sin x=\cdot 5$, $\cos x=\cdot 8660$ and

$$\Delta y=\cdot 8660\times\Delta x \text{ approximately.}$$

Suppose $\quad\Delta x=$ C.M. of $1^\circ=\cdot 0175$,

then
$$\begin{aligned}\Delta y&=\cdot 8660\times\cdot 0175 \text{ approximately}\\&=\cdot 0152 \text{ approximately,}\end{aligned}$$

i.e. if the angle increases from 30° to 31° the increase in the sine is approximately ·0152, or

$$\sin 31^\circ=\cdot 5152 \text{ approximately.}$$

[Actually $\sin 31^\circ=\cdot 5150$ correct to 4 figures.]

This might be stated thus:

$$f(x+h)=f(x)+hf'(x) \text{ approximately.}$$

Here
$$\left.\begin{aligned}f(x)&=\sin x\\ \therefore f'(x)&=\cos x\end{aligned}\right\}.$$

Also
$$x=\frac{\pi}{6},\quad h=\cdot 0175.$$

$$\begin{aligned}\therefore \sin 31^\circ&=\sin 30^\circ+\cdot 0175\times\cos\frac{\pi}{6}\\&=\cdot 5000+\cdot 0175\times\cdot 8660, \text{ \&c.}\end{aligned}$$

Ex. 3. Find maximum and minimum values of

$$4\cos x+3\sin x.$$

Let
$$y=4\cos x+3\sin x.$$

$$\therefore \frac{dy}{dx}=-4\sin x+3\cos x. \quad \text{[Ex. 1, Exs. LVII.]}$$

If y is a maximum or minimum, $\frac{dy}{dx}=0$.

$$\therefore \tan x=\frac{3}{4}.$$

i.e. $x=$ C.M. of
$$\begin{array}{ll}36^\circ\ 52' & -143^\circ\ 08'\\ 216^\circ\ 52' & -323^\circ\ 08'\\ \text{\&c.} & \text{\&c.}\end{array}$$

Take for example $x = \text{C.M. of } (-143^\circ\ 08')$,

$$\begin{aligned} y &= 4\cos(-143^\circ\ 08') + 3\sin(-143^\circ\ 08') \\ &= -4\cos 36^\circ\ 52' - 3\sin 36^\circ\ 52' \\ &= -4\,.\,\frac{4}{5} - 3\,.\,\frac{3}{5} = -5. \end{aligned}$$

If $x = \text{C.M. of } (-142^\circ)$

$$\begin{aligned} y &= -4\cos 38^\circ - 3\sin 38^\circ \\ &= -3{\cdot}1520 - 1{\cdot}8471 \\ &= -4{\cdot}9991. \end{aligned}$$

If $x = \text{C.M. of } (-144^\circ)$

$$\begin{aligned} y &= -4\cos 36^\circ - 3\sin 36^\circ \\ &= -3{\cdot}2360 - 1{\cdot}7634 \\ &= -4{\cdot}9994. \end{aligned}$$

$\therefore\ -5$ corresponding to $x = \text{C.M. of } (-143^\circ\ 08')$ is a minimum value.

Or we might say, since

$$\frac{dy}{dx} = -4\sin x + 3\cos x,$$

$$\therefore\ \frac{d^2y}{dx^2} = -4\cos x - 3\sin x.$$

and when $x = \text{C.M. of } -143^\circ\ 08'$, $\cos x$ and $\sin x$ are both negative.

$$\therefore\ \frac{d^2y}{dx^2} \text{ is positive,}$$

and $x = \text{C.M. of } 143^\circ\ 08'$ gives a minimum value of y.

209. {This particular kind of problem is more easily solved by ordinary Trigonometry, for

$$4\cos x + 3\sin x = 5\sin(x + \alpha),$$

where α is the acute angle whose tangent is $\frac{4}{3}$, i.e. $53^\circ\ 8'$, and

$5\sin(x+\alpha)$ always lies between -5 and $+5$, having the value -5 when

$$\sin(x+\alpha) = -1,$$

i.e. $$x + 53^\circ\ 8' = 270^\circ,\ 630^\circ,\ \ldots,\ -90^\circ,\ -450^\circ,\ \ldots,$$

and having the value $+5$ when $\sin(x+\alpha) = +1$,

i.e. $$x + 53^\circ\ 8' = 90^\circ,\ 450^\circ,\ \ldots,\ -270^\circ,\ -630^\circ.\}$$

EXERCISES. LVII.

1. Prove that if $\mathbf{y = \cos x}$, $\dfrac{\mathbf{dy}}{\mathbf{dx}} = -\mathbf{\sin x}$.

Illustrate as in §§ 205, 206.

2. Get $\dfrac{d\sin x}{dx} = \cos x$ by taking it as

$$\underset{\Delta x \to 0}{\mathrm{Lt}} \frac{\sin\left(x + \frac{\Delta x}{2}\right) - \sin\left(x - \frac{\Delta x}{2}\right)}{\Delta x}.$$

3. Find the gradients of the curve

$$y = 2\sin x + 3\cos x$$

at the points where $x = 0, 1, 2, \dfrac{\pi}{6}, \dfrac{2\pi}{3}, \tan^{-1}\dfrac{2}{3}, \tan^{-1}\left(-\dfrac{3}{2}\right), \pi - \tan^{-1}\dfrac{3}{2}$.

Draw the graph of $y = 2\sin x + 3\cos x$ from $x = -\pi$ to $x = \pi$, and check your results.

Find the equations of the tangents to $y = 2\sin x + 3\cos x$ at the points where $x = 0, 1, \dfrac{2\pi}{3}, \tan^{-1}\left(-\dfrac{3}{2}\right), \tan^{-1}\dfrac{2}{3}$.

4. If $\mathbf{y = \sin nx}$ where n is a constant, prove $\dfrac{\mathbf{dy}}{\mathbf{dx}} = \mathbf{n\cos nx}$, and if $\mathbf{y = \cos nx}$, $\dfrac{\mathbf{dy}}{\mathbf{dx}} = -\mathbf{n\sin nx}$.

5. Write down $\dfrac{dy}{dx}$ for the following values of y:

(i) $\cos 5x$, (ii) $3\sin 2x$, (iii) $\dfrac{1}{4}\sin 4x - \dfrac{1}{3}\cos 3x$,

(iv) $a\sin nx + b\cos nx$, where a, b, n are constants.

6. The radius of a circle starts in the position OA and rotates with uniform angular velocity ω radians/sec. OP is its position at the end of t seconds. OB is the radius perpendicular to OA and PM, PN are perpendicular to OA, OB. If the radius is a, what are OM (x) and ON (y)?

What is the speed of M at the end of t seconds?
What is the acceleration of M at the end of t seconds?
What is the speed of N at the end of t seconds?
What is the acceleration of N at the end of t seconds?

In each case shew that the acceleration varies as the distance from O [you should get M's acceleration $= -\omega^2 x$], and that M oscillates backwards and forwards from A to A′ and back, performing a complete oscillation in $\frac{2\pi}{\omega}$ seconds. M is said to have **Simple Harmonic Motion.**

7. If x increases at a uniform rate, shew that the rate of increase of $\sin x$ decreases as x increases from 0 to $\frac{\pi}{2}$, i.e. that $\sin x$ increases faster when x is near zero and more slowly when x is near $\frac{\pi}{2}$.

[Look at the difference columns in the table of Natural Sines.]

8. Given $\sin 60° = \cdot 8660$ and $\cos 60° = \cdot 5000$, find approximately $\sin 61°$, $\sin 60° 30'$, $\cos 61°$, $\cos 60° 30'$.

9. Find $\frac{d^2y}{dx^2}$, (i) if $y = \sin x$, (ii) if $y = \cos x$.

In each case shew that $$\frac{d^2y}{dx^2} + y = 0.$$

10. Find $\frac{d^2y}{dx^2}$, (i) if $y = a \cos nx$, (ii) if $y = a \sin nx$.

In each case shew that $$\frac{d^2y}{dx^2} + n^2 y = 0.$$

11. Find maximum and minimum values of

$$\text{(i) } 2 \sin x + 3 \cos x, \quad \text{(ii) } 4 \cos x - 5 \sin x.$$

12. What is (i) $\int \sin x \,.\, dx$, (ii) $\int \cos x \,.\, dx$?

13. Find the area bounded by the x-axis and that portion of $y = \sin x$ which lies between $x = 0$ and $x = \pi$.

Draw a figure and be quite clear what the answer means, e.g. if 1″ represents $\frac{\pi}{2}$ along OX and 1″ represents 1 along OY, what is the area in square inches?

14 Find (i) $\int_0^1 \sin x \,.\, dx$, (ii) $\int_0^1 \cos x \,.\, dx$,

$$\text{(iii) } \int_0^{\frac{\pi}{2}} \sin x \,.\, dx, \quad \text{(iv) } \int_0^{\frac{\pi}{2}} \cos x \,.\, dx.$$

15. Find (i) $\int_0^{2\pi} \sin x \,.\, dx$, (ii) $\int_0^{2\pi} \cos x \,.\, dx$,

(iii) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin x \,.\, dx$, (iv) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \,.\, dx$.

Draw figures to explain the results.

16. What is (i) $\int \sin nx \,.\, dx$, (ii) $\int \cos nx \,.\, dx$?

[v. Ex. 4.]

17. Prove and draw figures to illustrate the following results:

$$\text{(i)} \int_0^{\pi} \sin x \,.\, dx = 2\int_0^{\frac{\pi}{2}} \sin x \,.\, dx = \int_0^{3\pi} \sin x \,.\, dx$$

$$= 2\int_0^{\frac{3\pi}{2}} \sin x \,.\, dx,$$

$$\text{(ii)} \int_0^{\frac{\pi}{2}} \sin x \,.\, dx = \int_0^{\frac{\pi}{2}} \cos x \,.\, dx,$$

$$\text{(iii)} \int_0^{\pi} \cos x \,.\, dx = 0 = \int_0^{2\pi} \cos x \,.\, dx = \int_0^{3\pi} \cos x \,.\, dx,$$

$$\text{(iv)} \int_0^{\pi} \sin x \,.\, dx = 2\int_0^{\frac{\pi}{2}} \sin 2x \,.\, dx = 3\int_0^{\frac{\pi}{3}} \sin 3x \,.\, dx.$$

18. Write down

(i) $\int \sin 2x \,.\, dx$, (ii) $\int \cos 2x \,.\, dx$,

(iii) $\int (3 \sin 2x + 4 \cos 3x)\, dx$, (iv) $\int (1 + \cos 2x)\, dx$,

(v) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2x \,.\, dx$, (vi) $\int_0^{\pi} \cos 2x \,.\, dx$,

(vii) $\int \sin x \cos x \,.\, dx$ [remember $\sin 2x = 2 \sin x \cos x$],

(viii) $\int \sin^2 x \,.\, dx$ [$\cos 2x = 1 - 2\sin^2 x$],

(ix) $\int \cos^2 x \,.\, dx$ [$\cos 2x = 2\cos^2 x - 1$],

(x) $\int_0^{\pi} \sin^2 x \,.\, dx$, (xi) $\int_0^{\pi} \cos^2 x \,.\, dx$.

What is the sum of (viii) and (ix)? How could you foretell this result without finding the separate integrals?

19. Find the C.G. of the area bounded by the x-axis and that portion of $y = \sin x$ that lies between $x = 0$ and $x = \pi$.

[If you try to find $\bar{x}$ by the ordinary method, you will meet with an integral which you do not know how to evaluate. How can you evade the difficulty?]

20. What is the value of $\int_0^{\pi} x \sin x \,.\, dx$?

21. Find

(i) $\int \sin 3x \cos 2x \,.\, dx$, (v) $\int \sin px \cos qx \,.\, dx$,

(ii) $\int \cos 3x \sin 2x \,.\, dx$, (vi) $\int \cos px \cos qx \,.\, dx$,

(iii) $\int \cos 3x \cos 2x \,.\, dx$, (vii) $\int \sin px \sin qx \,.\, dx$.

(iv) $\int \sin 3x \sin 2x \,.\, dx$,

22. In Ex. 2, § 208, what would you get if you used the approximation

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x)\,?$$

23. The area of a triangle is calculated from the statement that two sides are 2 feet and 3 feet and the included angle 40°. If the sides are measured correctly, but an error of 10′ is made in measuring the angle, find approximately in square inches the error in the area.

Use no tables. You are given $\sin 40° = \cdot 6428$; $\cos 40° = \cdot 7660$.

24. Find the area bounded by the curve $y = 3 \sin x + 2 \cos x$, the x-axis and the ordinates $x = 0$ and $x = \pi$.

25. Find the mean value of $\sin x$ between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$.

Check approximately by calculating $\frac{1}{16}[\sin 45° + \sin 46° + \ldots + \sin 60°]$.

26. The distance (s feet) of a particle moving in a straight line, from a fixed point in the line, at the end of t seconds is given by $s = a \cos t + b \sin t$. Shew that the acceleration is $-s$ ft./sec.^{2}

Find the speed and acceleration at the end of one second, if $a = 4$, $b = 3$.

27. Find the mean value of the ordinate of a semicircle when the ordinates are erected at equal intervals (i) along the arc, (ii) along the diameter.

Draw figures shewing 15 ordinates in each case.

Differential coefficient of tan x.

210. With same notation as in § 204

$$y = \tan x,$$

$$y + \Delta y = \tan(x + \Delta x).$$

$$\therefore \Delta y = \tan(x + \Delta x) - \tan x$$

$$= \frac{\sin(x+\Delta x)}{\cos(x+\Delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin \Delta x}{\cos(x+\Delta x)\cos x}.$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{\sin \Delta x}{\Delta x} \cdot \frac{1}{\cos(x+\Delta x)} \cdot \frac{1}{\cos x}.$$

When $\Delta x \rightarrow 0$, $\dfrac{\sin \Delta x}{\Delta x} \rightarrow 1$ and $\dfrac{1}{\cos(x+\Delta x)} \rightarrow \dfrac{1}{\cos x}$.

$$\therefore \mathbf{\frac{dy}{dx} = \frac{1}{\cos^2 x} = \sec^2 x} \left(= 1 + \tan^2 x\right).$$

211. Illustration of this result.

Angle in degrees	x = C.M. of angle	y = $\tan x$			
30°	·5235988	·5773503			
	$x+\Delta x$	$y+\Delta y$	Δx	Δy	$\dfrac{\Delta y}{\Delta x}$
31°	·5410521	·6008606			
30° 30′	·5323254	·5890450			
30° 12′	·5270894	·5820139			
30° 6′	·5253441	·5796797			
30° 1′	·5238897	·5777382			

Fill up the blank columns and shew that the last column comes nearer and nearer to $\sec^2 30°$.

Geometrical proof.

212. As in § 207, $\angle$ AOP $= x$ radians, $\angle$ AOQ $= x + \Delta x$ radians. OP, OQ meet the tangent at A in p, q. [Fig. 119.]

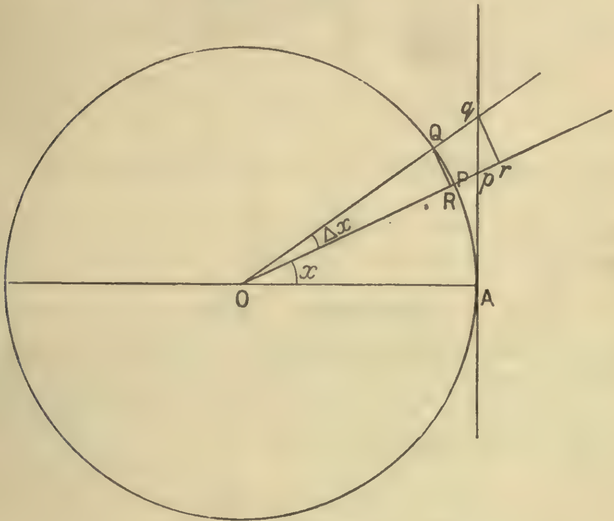


Fig. 119.

qr, QR are perpendicular to OP.

$$y = \tan x = \frac{\mathsf{A}p}{r}.$$

$$y + \Delta y = \tan(x + \Delta x) = \frac{\mathsf{A}q}{r}.$$

$$\therefore\ \Delta y = \frac{pq}{r},$$

and $$\Delta x = \frac{\widehat{\mathsf{PQ}}}{r}.$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{pq}{\overset{\frown}{PQ}}$$

$$= \frac{qr \,.\, \sec x}{QR} \cdot \frac{QR}{\overset{\frown}{PQ}} \cdot \quad (\angle pqr = x.)$$

Now $$\frac{qr}{QR} = \frac{Oq}{OQ} = \frac{Oq}{OA} = \sec(x + \Delta x).$$

$$\therefore \frac{\Delta y}{\Delta x} = \sec x \,.\, \sec(x + \Delta x) \,.\, \frac{QR}{\overset{\frown}{PQ}} \,.$$

As $\Delta x \to 0$, $\sec(x + \Delta x) \to \sec x$ and $\dfrac{QR}{\overset{\frown}{PQ}} \to 1$.

$$\therefore \mathbf{\frac{dy}{dx} = sec^2\, x.}$$

EXERCISES. LVIII.

1. Draw $y = \tan x$ from $x = 0$ to $x = 1{\cdot}2$ and verify that the gradient is $\sec^2 x$ at a few points.

2. Draw a sketch of $y = \tan x$ from $x = -2\pi$ to $x = +2\pi$. How can you see that $\dfrac{dy}{dx}$ is positive for all values of x?

3. (i) **If $y = \cot x$, prove $\dfrac{dy}{dx} = -\operatorname{cosec}^2 x$.**

(ii) **If $y = \sec x$, prove $\dfrac{dy}{dx} = \sec x \tan x$.**

(iii) **If $y = \operatorname{cosec} x$, prove $\dfrac{dy}{dx} = -\operatorname{cosec} x \cot x$.**

4. If $y = \tan nx$, prove $\dfrac{dy}{dx} = n \sec^2 nx$, and find $\dfrac{dy}{dx}$ (i) if $y = \cot nx$, (ii) if $y = \sec nx$, (iii) if $y = \operatorname{cosec} nx$.

5. Write down:

(i) $\int \sec^2 x \,.\, dx$, (ii) $\int \operatorname{cosec}^2 x \,.\, dx$,

(iii) $\int \sec x \tan x \,.\, dx$, (iv) $\int \operatorname{cosec} x \cot x \,.\, dx$,

(v) $\int (1 + \tan^2 x)\, dx$, (vi) $\int (1 + \cot^2 x)\, dx$,

(vii) $\int \frac{\sin x}{\cos^2 x}\,dx$, (viii) $\int \frac{\cos x}{\sin^2 x}\,dx$,

(ix) $\int \sec^2 3x \,.\, dx$, (x) $\int \tan^2 x \,.\, dx$,

(xi) $\int \tan^2 3x \,.\, dx$.

6. If
$$y = \sec x + \tan x,$$
shew that
$$\frac{1}{y} \cdot \frac{dy}{dx} = \sec x.$$

7. Knowing $\tan 45° = 1$, and $\sec 45° = \sqrt{2}$, find approximately $\tan 45{\cdot}1°$.

8. AOB is a quadrant of a circle, centre O. BX is the tangent at B.

A point T travels along BX and in every position of T, TO is joined cutting the circumference in P.

If $\angle$ BOP $= \theta$, shew that the

speed of P = speed of T × $\cos^2\theta$.

If the speed of T be uniform—5000 ft./sec.—find the speed of P when $\theta =$ (i) 45°, (ii) 60°, (iii) 80°, (iv) 89°, (v) $89\frac{1}{2}°$, (vi) $89\frac{3}{4}°$. (v. § 37.)

Also if OB be 10 ft. and OP have a uniform angular velocity of 1 radian per minute, find in ft./sec. the speed of T for the same values of θ.

9. An electric current is measured by a tangent galvanometer, the current being proportional to the tangent of the deflection. If the deflection is read as 45° and an error of 1 % is made in reading it, shew that the percentage error in the current is approximately $\frac{\pi}{2}$.

10. If x is the deflection in a tangent galvanometer and a given small error is made in reading the deflection, shew that the percentage error in the current is proportional to $(\tan x + \cot x)$.

213. *Ex.* A man in a boat 6 miles from shore wishes to reach a village 14 miles distant from the point of the shore nearest to him. He can walk 4 miles an hour and row 3 miles an hour. Where should he land in order to reach the village in the least possible time? What will this least time be?

[The shore is supposed straight.]

B is the boat. V the village. NV the shore. BN perpendicular to NV. P any point in NV. Call the angle NBP θ. [Fig. 120.]

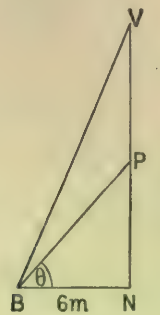


Fig. 120.

Then $$\text{BP} = 6 \sec \theta,$$

and $$\text{PV} = 14 - 6 \tan \theta.$$

$\therefore$ Total time taken if he lands at P is

$$\left(\frac{6 \sec \theta}{3} + \frac{14 - 6 \tan \theta}{4}\right) \text{ hours.}$$

Call this t, so that

$$t = 2 \sec \theta - \frac{3}{2} \tan \theta + \frac{7}{2}.$$

$$\therefore \frac{dt}{d\theta} = 2 \sec \theta \tan \theta - \frac{3}{2} \sec^2 \theta.$$

For a minimum value of t, $\dfrac{dt}{d\theta} = 0$.

$$\therefore \sec \theta \left(2 \tan \theta - \frac{3}{2} \sec \theta\right) = 0.$$

$$\therefore 2 \frac{\sin \theta}{\cos \theta} - \frac{3}{2 \cos \theta} = 0,$$

since $\sec \theta \neq 0$.

$$\therefore \sin\theta = \frac{3}{4}.$$

$$\therefore \sec\theta = \frac{4}{\sqrt{7}},$$

and
$$\tan\theta = \frac{3}{\sqrt{7}}.$$

$$\therefore \mathsf{NP} = 6\tan\theta = \frac{18}{\sqrt{7}} = 6{\cdot}804,$$

and
$$t = \frac{8}{\sqrt{7}} - \frac{9}{2\sqrt{7}} + \frac{7}{2} = \frac{7+\sqrt{7}}{2} = 4{\cdot}823.$$

[Shew that this is a minimum value.]

$\therefore$ He must land 6·804 miles from N and his time will be 4·823 hours.

EXERCISES. LIX.

1. A wall 6 feet high runs parallel to and 5 feet from another wall. Find the length of the shortest ladder that will reach from the ground to the second wall over the first.

2. An elastic string has one end fixed at A and the other B moves along OY perpendicular to OA.

If OA = 2 feet, find at what rate the other end is moving when $\angle$OAB = 60°, supposing the angle OAB to be increasing at a uniform rate of 1° per second.

3. With the same string as in Ex. 2, find the rate at which $\angle$OAB is increasing when $\angle$OAB = 60°, supposing B to be moving at 1 inch per second.

4. A man 6 feet high walks at 6 feet per second away from a lamp-post 10 feet high.

Find the rate at which his shadow is increasing. If θ be the angle made with the ground by the line joining the top of his head to the top of the lamp-post when he is x feet from the post, prove $x = 4\cot\theta$. Hence find the rate at which θ is decreasing when he is 8 feet from the lamp.

CHAPTER X

PRODUCT. QUOTIENT. FUNCTION OF FUNCTION. INVERSE FUNCTION. IMPLICIT FUNCTIONS

Differential coefficient of a product.

214. SUPPOSE $y = x^2 \sin x$.

Here y is the product of two functions, viz. x^2 and $\sin x$, each of which we can differentiate separately.

$$y + \Delta y = (x + \Delta x)^2 \sin (x + \Delta x).$$

$$\therefore \Delta y = x^2 \{\sin (x + \Delta x) - \sin x\} + 2x\Delta x \,.\, \sin (x + \Delta x) + (\Delta x)^2 \sin (x + \Delta x).$$

$$\therefore \frac{\Delta y}{\Delta x} = x^2 \,.\, \frac{\sin (x + \Delta x) - \sin x}{\Delta x} + 2x \sin (x + \Delta x) + \Delta x \,.\, \sin (x + \Delta x).$$

Now as $\Delta x \to 0$, $\dfrac{\sin (x + \Delta x) - \sin x}{\Delta x} \to \cos x$, $\sin (x + \Delta x) \to \sin x$, and $\Delta x \to 0$.

$$\therefore \frac{dy}{dx} = x^2 \cos x + 2x \sin x.$$

215. We might have arranged this as follows, so as to shew more clearly the significance of the different components of the result.

$$\Delta y = (x+\Delta x)^2 \sin(x+\Delta x) - (x+\Delta x)^2 \sin x + (x+\Delta x)^2 \sin x - x^2 \sin x,$$

where the term $(x+\Delta x)^2 \sin x$ has been added and subtracted as a link between $(x+\Delta x)^2 \sin(x+\Delta x)$ and $x^2 \sin x$.

$$\therefore \frac{\Delta y}{\Delta x} = (x+\Delta x)^2 \frac{\sin(x+\Delta x) - \sin x}{\Delta x} + \sin x \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \mathsf{A}(x+\Delta x)^2 + \mathsf{B} \,.\, \sin x, \text{ say.}$$

Now, as $\Delta x \to 0$, $\mathsf{A} \to \dfrac{d \sin x}{dx}$, i.e. $\cos x$, and $\mathsf{B} \to \dfrac{d\,.\,x^2}{dx}$, i.e. $2x$. Also $(x+\Delta x)^2 \to x^2$.

$$\therefore \frac{dy}{dx} = x^2 \cos x + 2x \sin x,$$

or
$$\frac{d(x^2 \sin x)}{dx} = x^2 \,.\, \frac{d \sin x}{dx} + \sin x \,.\, \frac{dx^2}{dx}.$$

EXERCISES. LX.

1. Go through the work again, using as a link

$$x^2 \sin(x+\Delta x).$$

2. Find in the same way $\dfrac{dy}{dx}$,

(i) when $y = (2x+3)\cos x$,

(ii) when $y = x^3 \tan x$.

216. In the general case let $y = uv$, where u and v are functions of x.

If x receive a small increment Δx, this will produce small increments Δu, Δv in u and v and these will produce a small increment Δy in y.

Thus
$$y + \Delta y = (u+\Delta u)(v+\Delta v)$$
$$= uv + u \,.\, \Delta v + v \,.\, \Delta u + \Delta u \,.\, \Delta v.$$
$$\therefore \Delta y = (u+\Delta u)\Delta v + v\Delta u.$$
$$\therefore \frac{\Delta y}{\Delta x} = (u+\Delta u) \,.\, \frac{\Delta v}{\Delta x} + v \,.\, \frac{\Delta u}{\Delta x}.$$

As $\Delta x \to 0$, $(u+\Delta u) \to u$, $\dfrac{\Delta v}{\Delta x} \to \dfrac{dv}{dx}$ and $\dfrac{\Delta u}{\Delta x} \to \dfrac{du}{dx}$.

$$\therefore \mathbf{\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}.$$

217. Notice that this result may be stated as follows:

(i) Differentiate uv as if u were constant. This gives $u\,\dfrac{dv}{dx}$.

(ii) Differentiate uv as if v were constant. This gives $v.\dfrac{du}{dx}$.

(iii) Add these together.

If we divide both sides by y or uv, we get

$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{u}.\frac{du}{dx} + \frac{1}{v}.\frac{dv}{dx}.$$

218. *Ex.* 1. If $\qquad y = x^5 \times x^8.$

Then by the formula

$$\frac{dy}{dx} = x^5 \times 8x^7 + x^8 \times 5x^4$$

$$= 13x^{12},$$

which agrees with what we get by writing

$$y = x^{13}$$

and differentiating in the ordinary way.

Ex. 2. If $\qquad y = \cos x \,.\, \tan x,$

$$\frac{dy}{dx} = \cos x \,.\, \sec^2 x + \tan x\,(-\sin x)$$

$$= \frac{1}{\cos x} - \frac{\sin^2 x}{\cos x}$$

$$= \frac{\cos^2 x}{\cos x} = \cos x,$$

which agrees with what we get by writing

$$y = \sin x \left[= \cos x \times \frac{\sin x}{\cos x} \right].$$

Ex. 3. If $y = \sin^2 x = \sin x \,.\, \sin x$,

$$\frac{dy}{dx} = \sin x \,.\, \cos x + \sin x \,.\, \cos x$$
$$= 2 \sin x \cos x.$$

EXERCISES. LXI.

Find $\frac{dy}{dx}$ for the following values of y:

1. $x \sin x$. **2.** $x^3 \tan x$.

3. $(3x^2 + 2x + 1)(2x + 3)$, (i) by treating as a product, (ii) by multiplying out.

4. $2 \sin x \cos x$. [Cf. Exs. LVII. 4.]

5. $x^{\frac{3}{2}} \times x^{-\frac{4}{5}}$, (i) by treating as a product, (ii) by writing as one term.

6. $\tan x \,.\, \operatorname{cosec} x$, (i) by treating as a product, (ii) by writing it $\sec x$.

7. $4x^5 \sin x$. **8.** $5x^2 \cos 3x$. **9.** $\cos^2 x$.

10. (i) $\cos^2 5x$, (ii) $\sin^2 5x$.

What is the sum of these two results? How can you see that this must be so without actually finding them?

11. $\sqrt{x} \cos x$. **12.** $\frac{\sin x}{x^2}$.

Product of more than two functions.

219. Suppose $y = uvw$, where u, v, w are all functions of x. uv is a function of x. Call it z.

$$\therefore\ y = zw.$$

$\therefore$ By our rule $$\frac{dy}{dx} = z \,.\, \frac{dw}{dx} + w \,.\, \frac{dz}{dx};$$

but since $$z = uv,$$

$$\therefore\ \frac{dz}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

$$\therefore\ \frac{dy}{dx} = uv \,.\, \frac{dw}{dx} + w\left(u\frac{dv}{dx} + v\frac{du}{dx}\right)$$

$$= \mathbf{vw\frac{du}{dx} + wu\frac{dv}{dx} + uv\frac{dw}{dx}}.$$

Ex. Prove similarly that if y is the product of four functions, say $uvwz$, then

$$\frac{dy}{dx} = vwz\frac{du}{dx} + wzu\frac{dv}{dx} + zuv\frac{dw}{dx} + uvw\frac{dz}{dx}.$$

220. If we divide both sides of the result in § 219 by y or uvw, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{u}\cdot\frac{du}{dx} + \frac{1}{v}\cdot\frac{dv}{dx} + \frac{1}{w}\cdot\frac{dw}{dx},$$

and similarly if $y = uvwz$,

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{u}\cdot\frac{du}{dx} + \frac{1}{v}\cdot\frac{dv}{dx} + \frac{1}{w}\cdot\frac{dw}{dx} + \frac{1}{z}\cdot\frac{dz}{dx}.$$

[Another way of proving these results will be seen later, § 263.]

Ex. If $y = (x+1)(2x+3)(3x+5)(4x+7)$,

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x+1} + \frac{2}{2x+3} + \frac{3}{3x+5} + \frac{4}{4x+7},$$

or

$$\frac{dy}{dx} = (x+1)(2x+3)(3x+5)(4x+7)\left[\frac{1}{x+1} + \frac{2}{2x+3} + \frac{3}{3x+5} + \frac{4}{4x+7}\right].$$

EXERCISES. LXII.

1. If $y = x^2(x+1)(x+3)$, find $\frac{dy}{dx}$:

(i) by treating y as the product of x^2, $(x+1)$, $(x+3)$,

(ii) by treating y as the product of x^2 and x^2+4x+3,

(iii) by taking $y = x^4 + 4x^3 + 3x^2$.

2. If $y = \frac{(x+1)(x+2)}{x}$, find $\frac{dy}{dx}$:

(i) by treating y as the product of $(x+1)$, $(x+2)$, $\frac{1}{x}$,

(ii) by taking $y = x + 3 + \frac{2}{x}$.

3. If $y = x^2 \sin x \cos x$, find $\frac{dy}{dx}$.

4. If $y=(2x^2+3x+1)^3$, find $\dfrac{dy}{dx}$.

5. If u is any function of x, find

(i) $\dfrac{du^2}{dx}$, (ii) $\dfrac{du^3}{dx}$, (iii) $\dfrac{du^4}{dx}$ in terms of u and $\dfrac{du}{dx}$.

Differential coefficient of a quotient.

221. Suppose $y=\dfrac{\sin x}{x^2}$.

Then
$$y+\Delta y=\frac{\sin(x+\Delta x)}{(x+\Delta x)^2}.$$

$$\therefore\ \Delta y=\frac{\sin(x+\Delta x)}{(x+\Delta x)^2}-\frac{\sin x}{x^2}$$
$$=\frac{x^2\sin(x+\Delta x)-(x+\Delta x)^2\sin x}{x^2(x+\Delta x)^2}$$
$$=\frac{x^2\{\sin(x+\Delta x)-\sin x\}-\sin x\{(x+\Delta x)^2-x^2\}}{x^2(x+\Delta x)^2},$$

where $x^2\sin x$ has been added and subtracted as a link between $x^2\sin(x+\Delta x)$ and $(x+\Delta x)^2\sin x$.

$$\therefore\ \frac{\Delta y}{\Delta x}=\frac{\mathrm{A}x^2-\mathrm{B}\sin x}{x^2(x+\Delta x)^2},$$

where A stands for $\dfrac{\sin(x+\Delta x)-\sin x}{\Delta x}$,

and B stands for $\dfrac{(x+\Delta x)^2-x^2}{\Delta x}$.

Now the limits of A and B as $\Delta x\to 0$ are $\cos x$ and $2x$ respectively [the differential coefficients of $\sin x$ and x^2], and the limit of $(x+\Delta x)^2$ is x^2.

$$\therefore\ \frac{dy}{dx}=\frac{x^2\cos x-2x\sin x}{x^4}=\frac{x\cos x-2\sin x}{x^3}$$

$$=\frac{x^2\cdot\dfrac{d(\sin x)}{dx}-\sin x\cdot\dfrac{d(x^2)}{dx}}{(x^2)^2}.$$

222. Generally, if $\quad y = \frac{u}{v},$

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}.$$

$$\therefore \Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$$

$$= \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}.$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v(v + \Delta v)},$$

and as $\Delta x \to 0$ the limits of $\frac{\Delta u}{\Delta x}$, $\frac{\Delta v}{\Delta x}$, $(v + \Delta v)$ are $\frac{du}{dx}$, $\frac{dv}{dx}$ and v respectively.

$$\therefore \mathbf{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}.$$

Ex. 1. Take $\quad y = \frac{4x^7}{3x^3},$

$$\frac{dy}{dx} = \frac{3x^3 \times 28x^6 - 4x^7 \times 9x^2}{9x^6}$$

$$= \frac{48x^9}{9x^6} = \frac{16}{3}x^3,$$

which agrees with what we get by taking $y = \frac{4}{3}x^4$.

Ex. 2. $\quad y = \frac{\sin x}{\cos x},$

$$\frac{dy}{dx} = \frac{\cos x \times \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x,$$

which agrees with what we have previously determined as the differential coefficient of $\tan x$.

EXERCISES. LXIII.

Find $\frac{dy}{dx}$ for the following values of y:

1. $\frac{x+1}{x+2}$. **2.** $\frac{x}{x^2+1}$.

3. $\frac{\sin x}{x}$ $\left[\text{(i) quotient, (ii) } \sin x \times \frac{1}{x}\right]$.

4. $\frac{x}{\sin x}$ [(i) quotient, (ii) $x \operatorname{cosec} x$].

5. $\frac{x^2+3x+1}{x^2-5x+2}$. **6.** $\frac{1}{x^2+3x+1}$.

7. $\frac{1-\sin x}{1+\sin x}$. **8.** $\frac{1}{\cos x}$.

9. $\frac{1}{u}$, where u is any function of x.

Function of a function.

223. The only functions of x which we have yet dealt with are powers and roots of x, $\sin x$, $\cos x$, &c. and products of these, and we have shewn how to find the rate at which any one of these changes with respect to x.

For instance we have

$$\frac{d \sin x}{dx} = \cos x,$$

which we may read:

The differential coefficient of the sine of any angle with respect to the angle itself = the cosine of the angle,

or, the rate of change of the sine of an angle per unit increase in the angle = cosine of the angle, it being understood that by "angle" we mean "number of radians in the angle."

We may write the above

$$\frac{d \,.\, \sin (\text{angle})}{d \,.\, \text{angle}} = \text{cosine (angle)}.$$

Thus
$$\frac{d\,.\sin 4x}{d\,.\,4x} = \cos 4x,$$
$$\frac{d\,.\sin(8x^2+3x)}{d\,.\,(8x^2+3x)} = \cos(8x^2+3x),$$
&c.

Similarly, from the result
$$\frac{d\,.\,x^3}{dx} = 3x^2,$$
we have
$$\frac{d\,.\,(2x+3)^3}{d\,.\,(2x+3)} = 3\,.\,(2x+3)^2,$$
$$\frac{d\,.\sin^3 x}{d\,.\sin x} = 3\sin^2 x.$$

EXERCISES. LXIV.

Write down

1. $\dfrac{d\,.\,(x^3)^4}{d\,.\,x^3}$. 2. $\dfrac{d\,.\,x^6}{d\,.\,x^2}$. 3. $\dfrac{d\,.\sqrt{\sin x}}{d\,.\sin x}$.

4. $\dfrac{d\,.\tan 4x}{d\,.\,4x}$. 5. $\dfrac{d\,.\sec^6 x}{d\,.\sec x}$. 6. $\dfrac{d\,.\dfrac{1}{2x+3}}{d\,.\,2x+3}$.

7. $\dfrac{d\,.\cos^2(4x+7)}{d\,.\cos(4x+7)}$. 8. $\dfrac{d\,.\,(4x+5)}{d\,.\,4x}$.

9. $\dfrac{d\,.\sqrt{x^2+x+1}}{d\,.\,(x^2+x+1)}$. 10. $\dfrac{d\,.\,(2\sin x+3\cos x)^{\frac{5}{2}}}{d\,.\,(2\sin x+3\cos x)}$.

224. Now suppose we have a function $y=\sin 3x$, and we want $\dfrac{dy}{dx}$. We do not get $\cos 3x$, for the fact corresponding to $\dfrac{d\sin x}{dx}=\cos x$ is not $\dfrac{d\sin 3x}{dx}=\cos 3x$, but $\dfrac{d\sin 3x}{d(3x)}=\cos 3x$, i.e. $\sin 3x$ is increasing $\cos 3x$ times as fast as $3x$,

but
$$\frac{d(3x)}{dx} = 3.$$

i.e. $3x$ is increasing 3 times as fast as x.

$\therefore$ combining these two statements

sin $3x$ is increasing $3\cos 3x$ times as fast as x,

or
$$\frac{\mathbf{d}\,.\sin 3\mathbf{x}}{\mathbf{dx}} = 3\cos 3\mathbf{x}$$

$$\left[\frac{d\,.\sin 3x}{dx} = \frac{d\,.\sin 3x}{d\,.\,3x} \times \frac{d\,.\,3x}{dx}\right].$$

225. As another example take $y = \sqrt{\tan x}$.

If $y = \sqrt{x}$, we know that $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, but we must not say that if $y = \sqrt{\tan x}$, $\frac{dy}{dx} = \frac{1}{2\sqrt{\tan x}}$, for the statement corresponding to

$$\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}} \text{ is not } \frac{d\sqrt{\tan x}}{dx} = \frac{1}{2\sqrt{\tan x}},$$

but
$$\frac{d\sqrt{\tan x}}{\tan x} = \frac{1}{2\sqrt{\tan x}},$$

i.e. $\sqrt{\tan x}$ is increasing $\frac{1}{2\sqrt{\tan x}}$ times as fast as $\tan x$.

Now
$$\frac{d\tan x}{dx} = \sec^2 x,$$

i.e. $\tan x$ is increasing $\sec^2 x$ times as fast as x.

$\therefore$ $\sqrt{\tan x}$ is increasing $\frac{\sec^2 x}{2\sqrt{\tan x}}$ times as fast as x,

or
$$\frac{\mathbf{d}\,\sqrt{\tan \mathbf{x}}}{\mathbf{dx}} = \frac{\sec^2 \mathbf{x}}{2\sqrt{\tan \mathbf{x}}}$$

$$\left[\frac{d\sqrt{\tan x}}{dx} = \frac{d\,\sqrt{\tan x}}{d\tan x} \times \frac{d\tan x}{dx}\right].$$

EXERCISES. LXV.

By similar reasoning find $\frac{dy}{dx}$ for the following values of y:

1. $\cos 5x$. **2.** $\tan 3x$. **3.** $\sin^3 x$ [i.e. $(\sin x)^3$].

4. $\sin(x^2+3)$. **5.** $\sin\sqrt{x}$. **6.** $(2x+3)^7$.

7. $(3x^2+5x+1)^4$. **8.** $\sin^3(7x+5)$.

226. We might have presented the argument of § 224 somewhat differently.

$$y = \sin u, \text{ where } u = 3x.$$

Now suppose y to be any function of u, and u any function of x.

If x receive a small increment Δx, this will produce a small increment Δu in u, and this increment in u will produce a small increment Δy in y.

Now
$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x},$$
however small Δx, Δu, Δy are.

But as $\Delta x \to 0$ $\frac{\Delta y}{\Delta x}, \frac{\Delta y}{\Delta u}, \frac{\Delta u}{\Delta x} \to \frac{dy}{dx}, \frac{dy}{du}, \frac{du}{dx}$ respectively.

$$\therefore \frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\mathbf{dy}}{\mathbf{du}} \cdot \frac{\mathbf{du}}{\mathbf{dx}}.$$

But in this case $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 3$.

$$\therefore \frac{dy}{dx} = 3\cos u$$
$$= 3\cos 3x.$$

Ex. 1. $y = \sin^3 x$.

Put $y = u^3$ where $u = \sin x$.

$$\frac{dy}{du} = 3u^2, \quad \frac{du}{dx} = \cos x.$$

$$\therefore \frac{dy}{dx} = 3u^2 \cos x$$

$$= 3 \sin^2 x \cos x.$$

This might be written

$$\frac{d(\sin^3 x)}{dx} = \frac{d(\sin^3 x)}{d(\sin x)} \times \frac{d(\sin x)}{dx}$$

$$= 3 \sin^2 x \times \cos x.$$

Ex. 2. $y = \sqrt{3x+4}$.

Put $y = \sqrt{u}$ where $u = 3x + 4$.

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}}, \quad \frac{du}{dx} = 3.$$

$$\therefore \frac{dy}{dx} = \frac{3}{2\sqrt{u}} = \frac{3}{2\sqrt{3x+4}}.$$

$$\left[\begin{aligned} \frac{d(\sqrt{3x+4})}{dx} &= \frac{d(\sqrt{3x+4})}{d(3x+4)} \times \frac{d(3x+4)}{dx} \\ &= \frac{1}{2\sqrt{3x+4}} \times 3. \end{aligned}\right]$$

227. Similarly $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dx}$.

Ex. 1. Suppose $y = \sqrt{\sin(4x^2 + 6x)}$.

We may say $y = \sqrt{u}$ where $u = \sin v$, where $v = 4x^2 + 6x$.

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}}, \quad \frac{du}{dv} = \cos v, \quad \frac{dv}{dx} = 8x + 6.$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times \cos v \times (8x+6)$$

$$= \frac{(4x+3)\cos(4x^2+6x)}{\sqrt{\sin(4x^2+6x)}},$$

$$\left[\text{or}\quad \frac{dy}{dx}=\frac{d\sqrt{\sin(4x^2+6x)}}{d\sin(4x^2+6x)}\times\frac{d\sin(4x^2+6x)}{d(4x^2+6x)}\times\frac{d(4x^2+6x)}{dx}\right.$$

$$\left.=\frac{1}{2\sqrt{\sin(4x^2+6x)}}\times\cos(4x^2+6x)\times(8x+6).\right]$$

Ex. 2. $y=\sin^4(3x+4)$.

Put $y=u^4$ where $u=\sin(3x+4)$.

Put $u=\sin v,\quad v=3x+4.$

$$\frac{dy}{du}=4u^3,\qquad \frac{du}{dv}=\cos v,\qquad \frac{dv}{dx}=3.$$

$$\therefore\ \frac{dy}{dx}=4u^3\times\cos v\times 3$$

$$=12\sin^3(3x+4)\cos(3x+4),$$

$$\left[\text{or}\quad \frac{d\sin^4(3x+4)}{d\sin(3x+4)}\times\frac{d\sin(3x+4)}{d(3x+4)}\times\frac{d(3x+4)}{dx}\right.$$

$$\left.=4\sin^3(3x+4)\times\cos(3x+4)\times 3.\right]$$

EXERCISES. LXVI.

Find $\frac{dy}{dx}$ for the following values of y:

1. $\sqrt{\sin x}$. **2.** $\sin\sqrt{x}$. **3.** $\sin^4 x$.

4. $\sin 4x$. **5.** $\sin(x^4)$. **6.** $\sin^n x$.

7. $\sin nx$. **8.** $n\sin x$. **9.** $\cos^n x$.

10. $\cos nx$. **11.** $\sin(4x+5)$. **12.** $(3\sin x+4\cos x)^2$.

13. $\tan^2 x$. **14.** $\sec^2 x$.

[Why are the results of (13) and (14) the same?]

15. $(a\sin x+b\cos x)^n$. **16.** $\tan^n x$. **17.** $\sec^n x$.

18. $\sqrt{\tan x}$. **19.** $\tan\sqrt{x}$. **20.** $\sqrt[3]{\sin x}$.

21. $\sqrt{4x^2+3x+1}$. **22.** $\sqrt{ax^2+bx+c}$.

23. $(6x+1)^{20}$. **24.** $\sqrt{a^2+x^2}$. **25.** $(3x^2+2x+1)^{10}$.

26. $\dfrac{1}{(x^2+1)^5}$. **27.** $(ax+b)^n$. **28.** $\sqrt[3]{ax+b}$.

29. $\sin(\cos x)$. **30.** $\cos(\sin x)$. **31.** $\sqrt[4]{3x^2+2x+1}$.

32. $\dfrac{1}{\sqrt{1-x^2}}$. **33.** $\sin x^0$. **34.** $\sin(ax+b)$.

35. $\sqrt{\sin(ax+b)}$. **36.** $\sin^3(4x+5)$. **37.** $\sin^n(ax+b)$.

38. $\tan^n(ax+b)$. **39.** $\sec^n(ax+b)$.

40. $\sqrt{\sin(2x^2+5x+6)}$. **41.** $(a\sin^3 x+b\cos^3 x)$.

42. $(a\sin^3 x+b\cos^3 x)^5$. **43.** $\sqrt{3\sin^2 x-4\cos^2 x}$.

44. $\cos^5 3x$. **45.** $\tan^3(x^2)$.

46. $\sin(a+bx^n)$. **47.** $\sqrt{5\tan 2x+3\sec^2 2x}$.

48. $\sin^m(x^n)$. **49.** $3\sin x-4\sin^3 x$.

50. $4\cos^3 x-3\cos x$.

[In Exs. 51—60, u denotes any function of x.]

51. $\sin u$. **52.** $\sin^3 u$. **53.** u^2. **54.** u^n.

55. $\sqrt{u}$. **56.** $\dfrac{1}{u}$. **57.** $\sqrt{\sin u}$.

58. $\cos^n u$. **59.** $\sin\sqrt{u}$. **60.** $\sqrt{a^2+u^2}$.

Write down the values of $\int y\,dx$ corresponding to the following values of y:

61. $\sin^2 x\cos x$. **62.** $\sin^n x\cos x$. **63.** $\cos^n x\sin x$.

64. $\sin nx$. **65.** $\cos nx$. **66.** $\sec^2 nx$.

67. $\tan^2 nx$. **68.** $\sin(ax+b)$. **69.** $\cos(ax+b)$.

70. $(ax+b)^n$. **71.** $x(x^2+4)^5$. **72.** $\sin x^0$.

73. $\sec^5 x\,.\tan x$. **74.** $\dfrac{2x+3}{(x^2+3x+5)^2}$. **75.** $\dfrac{2x+3}{\sqrt{x^2+3x+5}}$.

228. By using the methods of §§ 218, 219, 222, 226 we may differentiate more complicated expressions.

Ex. 1. If $$y=\sqrt{\frac{1-x}{1+x}}, \text{ find } \frac{dy}{dx}.$$

We have $y=\sqrt{u}$, where $u=\dfrac{1-x}{1+x}$,

$$\frac{dy}{du}=\frac{1}{2\sqrt{u}} \text{ and } \frac{du}{dx}=-\frac{2}{(1+x)^2} \quad [\S\ 222]$$

$$\therefore \frac{dy}{dx}=-\frac{1}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}}.$$

Ex. 2. If $$y=\frac{x\sqrt{x^2-4}}{\sqrt{x^2-1}}, \text{ find } \frac{dy}{dx}.$$

1st method. $y=\dfrac{u}{v}$, where $u=x\sqrt{x^2-4}$, $v=\sqrt{x^2-1}$.

$$\therefore \frac{dy}{dx}=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}.$$

To get $\dfrac{du}{dx}$ we have $u=x\times\sqrt{x^2-4}$.

$$\therefore \frac{du}{dx}=\sqrt{x^2-4}+x\times\frac{d\sqrt{x^2-4}}{dx}$$

$$=\sqrt{x^2-4}+\frac{x^2}{\sqrt{x^2-4}}=\frac{2x^2-4}{\sqrt{x^2-4}},$$

and $$\frac{dv}{dx}=\frac{x}{\sqrt{x^2-1}}.$$

$$\therefore \frac{dy}{dx}=\frac{\dfrac{(2x^2-4)\sqrt{x^2-1}}{\sqrt{x^2-4}}-x.\sqrt{x^2-4}.\dfrac{x}{\sqrt{x^2-1}}}{x^2-1}=\frac{x^4-2x^2+4}{(x^2-4)^{\frac{1}{2}}(x^2-1)^{\frac{3}{2}}}.$$

2nd method. $y=xu$, where $u=\sqrt{\dfrac{x^2-4}{x^2-1}}$.

$$\therefore \frac{dy}{dx}=u+x\frac{du}{dx}.$$

Now $u=\sqrt{v}$, where $v=\dfrac{x^2-4}{x^2-1}$.

$$\therefore\ \frac{du}{dv}=\frac{1}{2\sqrt{v}} \text{ and } \frac{dv}{dx}=\frac{6x}{(x^2-1)^2}.$$

$$\therefore\ \frac{dy}{dx}=\sqrt{\frac{x^2-4}{x^2-1}}+x.\frac{3x}{(x^2-1)^2}\sqrt{\frac{x^2-1}{x^2-4}}$$

$$=\frac{x^4-2x^2+4}{(x^2-4)^{\frac{1}{2}}(x^2-1)^{\frac{3}{2}}}.$$

Ex. 3. $$y=\sin^3 2x.\cos^4 5x,$$

i.e. $y=uv$, where $u=\sin^3 2x$ and $v=\cos^4 5x$.

$$\therefore\ \frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx},$$

$$\frac{du}{dx}=\frac{d.\sin^3 2x}{d\sin 2x}.\frac{d\sin 2x}{d.2x}.\frac{d2x}{dx}$$

$$=3\sin^2 2x.\cos 2x.2$$

$$=6\sin^2 2x\cos 2x.$$

Similarly $$\frac{dv}{dx}=-20\cos^3 5x.\sin 5x.$$

$$\therefore\ \frac{dy}{dx}=6\sin^2 2x\cos 2x\cos^4 5x-20\sin^3 2x\cos^3 5x\sin 5x$$

$$=2\sin^2 2x\cos^3 5x(3\cos 2x\cos 5x-10\sin 2x\sin 5x)$$

$$=\sin^2 2x\cos^3 5x(13\cos 7x-7\cos 3x).$$

229. Notice that we could differentiate quotients without the special rule.

Take as an example the one first worked out

$$y=\frac{\sin x}{x^2}.$$

We could put this in the form

$$y=x^{-2}\times\sin x,$$

and treat it as a product.

$$\therefore \frac{dy}{dx} = x^{-2} \times \cos x + (-2x^{-3}) \sin x$$

$$= \frac{\cos x}{x^2} - \frac{2 \sin x}{x^3}$$

$$= \frac{x \cos x - 2 \sin x}{x^3}.$$

Ex. 2. $$y = \frac{\sin x}{\cos x},$$

ve could put this in the form

$$y = \sin x \times (\cos x)^{-1}$$

$$= u \times v,$$

where $u = \sin x$ and $v = (\cos x)^{-1}$.

We know $$\frac{du}{dx} = \cos x,$$

and $$\frac{dv}{dx} = \frac{d(\cos x)^{-1}}{d(\cos x)} \times \frac{d(\cos x)}{dx}$$

$$= \left(-\frac{1}{\cos^2 x}\right)(-\sin x)$$

$$= \frac{\sin x}{\cos^2 x}.$$

Now $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= \sin x \times \frac{\sin x}{\cos^2 x} + \frac{1}{\cos x} \times \cos x$$

$$= \tan^2 x + 1$$

$$= \sec^2 x.$$

EXERCISES. LXVII.

Find $\frac{dy}{dx}$ for the following values of y:

1. $\sqrt{\frac{1-x^2}{1+x^2}}$. 2. $x\sqrt{a^2-x^2}$. 3. $\sqrt{\frac{1-\sin^2 x}{1+\sin^2 x}}$.

4. $x^2 \cos^2 x$.

5. $(2x+1)^7 (3x+4)^5$.

6. $(x^2+x+1)^4 (2x^3-3x^2+4)^2$.

7. $\dfrac{x}{\sqrt{a^2-x^2}}$.

8. $\dfrac{\sqrt{1+x}}{1+\sqrt{x}}$.

9. $(ax+b)^m (cx+d)^n$.

10. $\sin^3 x \,.\, \sin 3x$.

11. $x(x^2+4)\sqrt{x^2-4}$.

12. $\sqrt{2\cos^3 5x - 3\sin^4 2x}$.

13. $\sin^n x \,.\, \sin nx$.

14. $\sin^n x \cos nx$.

15. $u^m v^n$, where u and v are functions of x.

16. $\sin^m u \cos^n v$.

17. $\sqrt{\dfrac{\sin^3 5x}{5+\cos 5x}}$.

18. $(x+\sqrt{a^2+x^2})^n$.

19. $\dfrac{n \sin x \sin nx}{\sin^n x}$.

20. $\sqrt[3]{\dfrac{u}{v}}$.

230. Sometimes y and x are each given as the function ot a third variable.

e.g. the co-ordinates (measured horizontally and vertically) at the end of t seconds, of a body projected with a velocity whose horizontal and vertical components are 30 and 40 ft./sec., are

$$x = 30t, \qquad y = 40t - 16t^2.$$

$$\therefore \frac{dx}{dt} = 30, \qquad \frac{dy}{dt} = 40 - 32t.$$

i.e. x is increasing 30 times as fast as t, and y is increasing $(40-32t)$ times as fast as t.

∴ y is increasing $\dfrac{40-32t}{30}$ times as fast as x,

or $$\frac{dy}{dx} = \frac{40-32t}{30} = \frac{dy}{dt}\Big/\frac{dx}{dt}.$$

From the given equations we might eliminate t. The first gives $t = \dfrac{x}{30}$, and substituting in the second, we get

$$y = 40 \times \frac{x}{30} - 16 \times \frac{x^2}{900}.$$

$$\therefore \frac{dy}{dx} = \frac{4}{3} - \frac{32x}{900}$$

$$= \frac{4}{3} - \frac{32 \times 30t}{900}$$

$$= \frac{40 - 32t}{30} \text{ as before,}$$

but in many cases the elimination of t would be inconvenient if not practically impossible.

The general case.

231. Suppose x and y are given in terms of a third variable t.

So that $\qquad x = f(t)$ and $y = \phi(t)$.

Suppose u receives a small increment Δt. This will produce small increments Δx and Δy in x and y.

Now $$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta t} \Big/ \frac{\Delta x}{\Delta t},$$

however small Δt, Δx, Δy are, but as Δt, and with it Δy and Δx, $\to 0$, $\frac{\Delta y}{\Delta x}, \frac{\Delta y}{\Delta t}, \frac{\Delta x}{\Delta t} \to \frac{dy}{dx}, \frac{dy}{dt}, \frac{dx}{dt}$ respectively.

$$\therefore \mathbf{\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}}.$$

EXERCISES. LXVIII.

1. If $y = 3t + 4t^2$ and $x = 2t + 1$, find $\frac{dy}{dx}$ in two ways.

2. If $x = 4 + 3t + 2t^3$ and $y = 7 + t^3$, find $\frac{dy}{dx}$.

3. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, find $\frac{dy}{dx}$.

4. Shew that the co-ordinates of any point of the parabola $y=\frac{x^2}{a}$ may be written in the form $x=am$, $y=am^2$.

Find $\frac{dy}{dx}$, (i) from $\frac{dy}{dm}$ and $\frac{dx}{dm}$, (ii) from $y=\frac{x^2}{a}$,

and shew that the results agree.

5. Shew that whatever θ is, $x=a\cos\theta$, $y=a\sin\theta$ satisfy the equation

$$x^2+y^2=a^2.$$

Prove $$\frac{dy}{dx}=-\cot\theta=-\frac{x}{y}.$$

Interpret this result geometrically.

Shew that the equation of the tangent at the point $a\cos\alpha$, $a\sin\alpha$ is

$$x\cos\alpha+y\sin\alpha=a.$$

6. Shew that whatever θ is, $x=a\cos\theta$, $y=b\sin\theta$ satisfy the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Prove $$\frac{dy}{dx}=-\frac{b}{a}\cot\theta=-\frac{b^2x}{a^2y}.$$

Hence shew that the equation of the tangent at the point $(a\cos\alpha, b\sin\alpha)$ is

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1.$$

7. The co-ordinates of a point on a cycloid are given by $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$.

Prove $$\frac{dy}{dx}=\cot\frac{\theta}{2}.$$

8. The co-ordinates of a point on an epicycloid are given by

$$\left.\begin{aligned} x&=(a+b)\cos\theta-b\cos\frac{a+b}{b}\theta \\ y&=(a+b)\sin\theta-b\sin\frac{a+b}{b}\theta \end{aligned}\right\}.$$

Prove $$\frac{dy}{dx}=\tan\frac{a+2b}{2b}\theta.$$

9. A body is projected from O with velocity u ft./sec. in a direction making an angle α with the horizontal. Taking as axes the horizontal and vertical through O in the plane of projection, the co-ordinates of the body at the end of t seconds are $\left(tu\cos\alpha,\ tu\sin\alpha-\frac{1}{2}gt^2\right)$. Prove that at the end of t seconds the direction of motion makes an angle $\tan^{-1}\left(\frac{u\sin\alpha-gt}{u\cos\alpha}\right)$ with the horizontal.

Hence find the time of reaching the greatest height.

232. The same problem might be presented in another form.

Ex. Find $\frac{d\sin x}{dx^3}$. i.e. find $\frac{du}{dv}$ where $u=\sin x$ and $v=x^3$.

We have
$$\frac{du}{dv}=\frac{du}{dx}\Big/\frac{dv}{dx}$$
$$=\frac{\cos x}{3x^2}.$$

Illustration of the meaning of this result.

Angle	x = C. M.	x^3	$\sin x$	
50°	·8726646	·664572	·7660444	
	$x+\Delta x$	$(x+\Delta x)^3$	$\sin(x+\Delta x)$	$\frac{\Delta(\sin x)}{\Delta(x^3)}$
51°	·8901179	·705251	·7771460	$\frac{·0111016}{·040679}=·2729$
50° 30′	·8813913	·684709	·7716246	$\frac{·0055802}{·020137}=·2771$
50° 10′	·8755735	·671242	·7679110	$\frac{·0018666}{·006670}=·2798$
50° 05′	·8741191	·667901	·7669785	$\frac{·0009341}{·003329}=·2806$

and $\frac{\cos x}{3x^2}=·2813.$

EXERCISES. LXIX.

Find:

1. $\frac{d(x^5)}{d(x^3)}$.

2. $\frac{d(\tan x)}{d(\sin x)}$.

3. $\frac{d(\sin 2x)}{d(\sin x)}$.

4. $\frac{d(2t^2-3t+1)}{d(3t^2+2t-1)}$.

5. $\frac{d\sin x}{d\cos x}$.

233. The device of expressing x and y in terms of some third variable is sometimes useful in integration.

Take the example in § 191 which we failed to do directly because we could not evaluate

$$\int_{-a}^{a} x^2\sqrt{a^2-x^2}\,dx.$$

If however we work in terms of θ ($\angle$ AOP) we have,

$$x = a\cos\theta.$$

$$\therefore\ \Delta x = -a\sin\theta\,.\,\Delta\theta \text{ app.}$$

and $$y = a\sin\theta.$$

[Δx and $\Delta\theta$ are of opposite signs.]

$$\therefore\ \text{Area of strip } \mathsf{P}pq\mathsf{Q} = 2a^2\sin^2\theta\,.\,\Delta\theta \text{ app.}$$

and $$\text{M.I.}\ \ ,,\ \ ,,\ \ ,, = 2ma^4\sin^2\theta\cos^2\theta\,\Delta\theta \text{ app.}$$

$$\therefore\ \text{M.I. of disc} = \int_0^{\pi} 2ma^4\sin^2\theta\cos^2\theta\,d\theta$$

$$=\frac{ma^4}{2}\int_0^{\pi}\sin^2 2\theta\,.\,d\theta = \frac{ma^4}{4}\int_0^{\pi}(1-\cos 4\theta)\,d\theta$$

$$=\frac{ma^4}{4}\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\pi} = \frac{m\pi a^4}{4} = \frac{\mathsf{M}a^2}{4}.\quad (\text{v. } \S\,280.)$$

EXERCISES. LXX.

1. Find the area of a quadrant of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad [\text{Put } x=a\cos\theta,\ y=b\sin\theta.]$$

2. The co-ordinates of a point on a cycloid are given by $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$, where θ is the angle turned through by the rolling circle.

Find the area between the x-axis and the portion of the curve traced out in one complete revolution of the rolling circle.

3. Find the M.I. of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(i) about the x-axis,

(ii) about the y-axis,

(iii) about a line through the origin perpendicular to the plane of the ellipse.

4. Find the resultant thrust on a circle, radius a ft., just immersed in water with its plane vertical. Also find the centre of pressure.

5. Find the C.G. of a quadrant of an ellipse.

Inverse functions.

234. If y is a function of x, then x is a function of y; for instance if $y=4x+3$, $x=\frac{y-3}{4}$, and generally if $y=f(x)$, x is some function of y, say $x=\phi(y)$.

In the above example $f(x)$ is $4x+3$ and $\phi(y)$ is $\frac{y-3}{4}$.

Notice that $$f\{\phi(x)\}=f\left(\frac{x-3}{4}\right)=4\,.\,\frac{x-3}{4}+3=x,$$
(v. § 87.)

and $$\phi\{f(x)\}=\phi(4x+3)=\frac{4x+3-3}{4}=x,$$

or the two operations denoted by f and ϕ neutralise one another.

$f(x)$ and $\phi(x)$ are called inverse functions. Sometimes $\phi(x)$ is written $f^{-1}(x)$.

e.g. $\sin x$ and $\sin^{-1}x$ are inverse functions

$$\sin(\sin^{-1}x)=x \text{ and } \sin^{-1}(\sin x)=x.$$

EXERCISES. LXXI.

1. If $y=\frac{2x-3}{4}$, find x in terms of y.

2. If $f(x)$ is $\frac{2x-3}{4}$, what is $f^{-1}(x)$?

Verify $$f\{f^{-1}(x)\}=x,$$
and $$f^{-1}\{f(x)\}=x.$$

235. If $\quad y = x^2, \quad x = \pm\sqrt{y}.$

Here y is said to be a single-valued function of x, i.e. given any value of x, there is only one corresponding value of y, but x is a double-valued function of y, i.e. given any value of y, there are two corresponding values of x.

e.g. if $x = 3$, $y = 9$; but if $y = 9$, $x = 3$ or -3.

In this case $f(x)$ is x^2, and $\phi(y)$ or $f^{-1}(y)$ is $\pm\sqrt{y}$ and one of the values of $\phi\{f(x)\}$ is x.

Differentiation of inverse functions.

236. If $\quad y = 4x + 3, \quad x = \dfrac{y-3}{4}.$

From the first of these relations we get $\dfrac{dy}{dx} = 4$, and from the second $\dfrac{dx}{dy} = \dfrac{1}{4} = \dfrac{1}{\dfrac{dy}{dx}}$.

EXERCISES. LXXII.

Verify that $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$ in each of the following cases:

1. $3x + 4y = 7$. **2.** $y = x^3$. **3.** $xy = k$. **4.** $y = \dfrac{3x+1}{4x-7}$.

237. We shall now prove the general theorem

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1.$$

The relation $\dfrac{\Delta y}{\Delta x} \times \dfrac{\Delta x}{\Delta y} = 1$ is always true however small Δx and Δy are. But by diminishing Δx and Δy indefinitely we can bring $\dfrac{\Delta y}{\Delta x}$ and $\dfrac{\Delta x}{\Delta y}$ as near as we like to $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$ respectively.

$$\therefore \frac{dy}{dx} \times \frac{dx}{dy} = 1.$$

This formula is of great use in differentiating inverse functions, for it often happens that $\dfrac{dx}{dy}$ is easier to get than $\dfrac{dy}{dx}$.

238. So long as y is a single-valued function of x, and x of y, this formula presents no difficulty.

Suppose $y = x^2$, so that $x = \pm\sqrt{y}$.

Then $$\frac{dy}{dx} = 2x \text{ and } \frac{dx}{dy} = \pm\frac{1}{2\sqrt{y}},$$

the sign being the same as in the line above.

i.e. if $$x = +\sqrt{y},\quad \frac{dx}{dy} = +\frac{1}{2\sqrt{y}},$$

if $$x = -\sqrt{y},\quad \frac{dx}{dy} = -\frac{1}{2\sqrt{y}},$$

so that in each case $$\frac{dx}{dy} = \frac{1}{2x},$$

and thus $$\mathbf{\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.}$$

Suppose for instance we want the value of $\frac{dy}{dx}$ when $x = 3$ [and therefore $y = 9$].

We have $$\frac{dy}{dx} = 2x,$$

so that when $x = 3$, $$\frac{dy}{dx} = 6.$$

Similarly, when $x = -3$, $\frac{dy}{dx} = -6$.

Suppose however we take $x = \pm\sqrt{y}$, then when $y = 9$, $x = \pm 3$,

and $$\frac{dx}{dy} = \pm\frac{1}{2\sqrt{y}},$$

so that when $y = 9$, $$\frac{dx}{dy} = \pm\frac{1}{6},$$

$$\left.\begin{array}{r}\frac{1}{6} \text{ corresponding to the case when } x = 3 \\ \text{and } -\frac{1}{6} \text{ corresponding to the case when } x = -3\end{array}\right\}.$$

∴ If in each case we are dealing with the same values of both x and y,

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1.$$

239. If we draw the graph of $y = x^2$, then $x = 3$ gives $y = 9$ and determines one point P_1 [Fig. 121],

$$\frac{dy}{dx} \text{ gives } \tan XT_1P_1,$$

$$\frac{dx}{dy} \text{ gives } \tan Yt_1P_1,$$

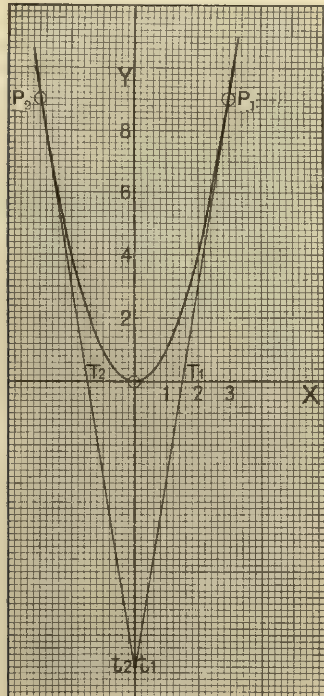


Fig. 121.

and $\tan XT_1P_1 . \tan Yt_1P_1 = 1$ since the angles are complementary: but if $y = 9$, $x = \pm 3$, and two points P_1 and P_2 are determined; $\frac{dy}{dx}$ gives $\tan XT_1P_1$ or $\tan XT_2P_2$ according as we take $x = 3$ or -3 and $\frac{dx}{dy}$ gives $\tan Yt_1P_1$ or $\tan Yt_2P_2$ (negative angle).

Now $$\left.\begin{array}{l}\tan XT_1P_1 \times \tan Yt_1P_1 = 1 \\ \tan XT_2P_2 \times \tan Yt_2P_2 = 1\end{array}\right\},$$

but $$\tan XT_1P_1 \times \tan Yt_2P_2 = -1$$

and $$\tan XT_2P_2 \times \tan Yt_1P_1 = -1 \Big\}.$$

i.e. $\frac{dy}{dx} \times \frac{dx}{dy} = 1$ if we deal with the same point throughout (either P_1 or P_2), but if we take $\frac{dy}{dx}$ with reference to the co-ordinates of P_1 and $\frac{dx}{dy}$ with reference to those of P_2 we do not get 1.

The inverse trigonometrical ratios.

240. Given $y = \sin^{-1} x$, to get $\frac{dy}{dx}$ we have

$$x = \sin y.$$

$$\therefore \frac{dx}{dy} = \cos y = \pm\sqrt{1 - x^2}.$$

$$\therefore \frac{dy}{dx} = \pm\frac{1}{\sqrt{1 - x^2}}.$$

The ambiguity of sign is accounted for as follows:

If we take $y = \sin^{-1} x$ to mean that y is some angle whose sine is x, then y is a many-valued function of x: for any value of x between -1 and 1, say $\cdot 866$, there is an infinite number of values of y, viz.:

$$\frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3} \dots, \quad -\frac{4\pi}{3}, -\frac{5\pi}{3}, -\frac{10\pi}{3}, -\frac{11\pi}{3} \dots,$$

and for some of these $\frac{dy}{dx}$ is +, for others − (see Fig. 122).

If we agree to understand by $\sin^{-1}x$ the angle between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ whose sine is x, then a glance at the figure shews that $\frac{dy}{dx}$ is always positive. [Fig. 122.]

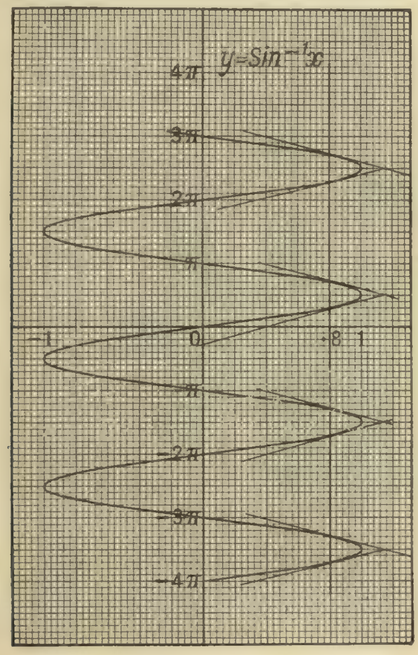


Fig. 122.

$\therefore$ **If $\mathbf{y = \sin^{-1} x}$, we take**

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

x	$y=\sin^{-1}x=$ C. M. of acute angle whose sine is ·5000.			
·5000	·5235988			
$x+\Delta x$	$y+\Delta y$	Δx	Δy	$\frac{\Delta y}{\Delta x}$
·5878	·6283	·0878	·1047	1·19+
·5592	·5934	·0592	·0698	1·18−
·5299	·5585	·0299	·0349	1·17
·5150381	·5410521	·0150381	·0174533	1·16
·5075384	·5323254	·0075384	·0087266	1·157

$$\frac{1}{\sqrt{1-x^2}}=\frac{1}{\sqrt{\frac{3}{4}}}=\sqrt{\frac{4}{3}}=\sqrt{1{\cdot}333}=\mathbf{1{\cdot}155}.$$

241. Similarly taking $\cos^{-1}x$ to mean the angle between 0 and π whose cosine is x, we can prove that if $\mathbf{y=\cos^{-1}x}$

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}.$$

Notice that since whatever x is,

$$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2},$$

$$\therefore\ \frac{d.\sin^{-1}x}{dx}+\frac{d.\cos^{-1}x}{dx}=0.$$

242. If $\qquad y=\tan^{-1}x,$

$$x=\tan y.$$

$$\therefore\ \frac{dx}{dy}=\sec^2 y=1+\tan^2 y=1+x^2.$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{1+x^2}.$$

Usually $\tan^{-1}x$ is taken to mean the angle between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ whose tangent is x, but the result just obtained shews that even though we make no such restriction, but take $\tan^{-1}x$ to mean any angle whose tangent is x, $\frac{dy}{dx}$ is always the same.

The graph shews that $\dfrac{dy}{dx}$ is always positive. [Fig. 123.]

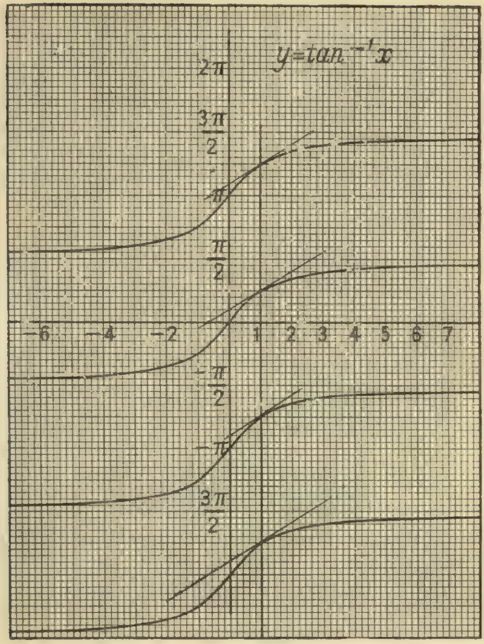


Fig. 123.

EXERCISES. LXXIII.

1. If $y=\cot^{-1} x$, prove $\dfrac{dy}{dx}=-\dfrac{1}{1+x^2}$.

2. If $y=\sec^{-1} x$, prove $\dfrac{dy}{dx}=\dfrac{1}{x\sqrt{x^2-1}}$.

3. If $y=\operatorname{cosec}^{-1} x$, prove $\dfrac{dy}{dx}=-\dfrac{1}{x\sqrt{x^2-1}}$.

Implicit functions.

243. It often happens that the relation between x and y is in such a form that it is impossible or undesirable to find y in terms of x.

e.g. we might have

$$2x^2 + xy - 3y^2 + x + 4y - 2 = 0 \quad \text{..............(1)}.$$

We can get $\frac{dy}{dx}$ without finding y in terms of x.

We must remember that

$$\frac{dy^2}{dx} = \frac{dy^2}{dy} \cdot \frac{dy}{dx} = 2y \cdot \frac{dy}{dx}.$$

Differentiate both sides of (1) with respect to x.

$$\therefore\ 4x + \left(x\frac{dy}{dx} + y\right) - 6y \cdot \frac{dy}{dx} + 1 + 4\frac{dy}{dx} = 0.$$

$$\therefore\ \frac{dy}{dx}(x - 6y + 4) + (4x + y + 1) = 0.$$

$$\therefore\ \mathbf{\frac{dy}{dx} = -\frac{4x + y + 1}{x - 6y + 4}}.$$

In this particular case we could have solved (1) as a quadratic in y.

Rearranging we get

$$3y^2 - (x + 4)\,y - (2x^2 + x - 2) = 0.$$

$$\therefore\ y = \frac{(x+4) \pm \sqrt{(x+4)^2 + 12\,(2x^2 + x - 2)}}{6}$$

$$= \frac{(x + 4) \pm \sqrt{25x^2 + 20x - 8}}{6}.$$

Taking the positive sign we have

$$\frac{dy}{dx} = \frac{1}{6}\left\{1 + \frac{50x + 20}{2\sqrt{25x^2 + 20x - 8}}\right\}$$

$$= \frac{1}{6}\left\{1 + \frac{25x + 10}{\sqrt{25x^2 + 20x - 8}}\right\}.$$

Now $$\sqrt{25x^2+20x-8}=6y-x-4.$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{6}\left\{1+\frac{25x+10}{6y-x-4}\right\}$$

$$=\frac{4x+y+1}{6y-x-4},$$

which agrees with the previous result. Similarly when we take the negative sign.

244. *Ex.* Find the equation of the tangent at the point (2, 3) on the curve

$$2x^2-4xy+7y^2=47.$$

We have $$4x-4\left(x\frac{dy}{dx}+y\right)+14y\frac{dy}{dx}=0.$$

$$\therefore\ \frac{dy}{dx}=\frac{4x-4y}{4x-14y}.$$

$$\therefore \text{ Gradient at } (2,\ 3)=\frac{8-12}{8-42}=\frac{2}{17}.$$

$$\therefore \text{ Equation of tangent is } y-3=\frac{2}{17}(x-2),$$

or $$\mathbf{2x-17y+47=0.}$$

EXERCISES. LXXIV.

1. If $x^2+y^2=a^2$, find $\frac{dy}{dx}$; and find the equations of the tangents at the points where $x=\frac{3}{5}a$.

Also find $\frac{dy}{dx}$ by taking $y=\sqrt{a^2-x^2}$.

2. If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, find $\frac{dy}{dx}$.

3. If $xy=k^2$, shew that $\frac{dy}{dx}=-\frac{y}{x}$. Hence shew that if the tangent at P meets the axes in Q and R,

$$PO=PQ=PR.$$

4. If $x^2y^3=k^5$, shew that $\dfrac{dy}{dx}=-\dfrac{2y}{3x}$.

Verify by writing $$y=\frac{k^{\frac{5}{3}}}{x^{\frac{2}{3}}}.$$

5. Find $\dfrac{dy}{dx}$ if $ax^2+2hxy+by^2=1$; a, h, b being constants.

6. Verify that $(2, -1)$ is a point on the curve
$$3x^2-4xy+2y^2-6x+7y-3=0,$$
and find the equation of the tangent to the curve at this point.

7. Find $\dfrac{dy}{dx}$ if $2\cos x+3\cos y=4$.

8. If $x^2+y^2-3x+10y-15=0$, find $\dfrac{dy}{dx}$. Use this to find the equations of the tangent and normal at $(4, -11)$ to the circle $x^2+y^2-3x+10y-15=0$. Verify that the normal passes through the centre.

9. Find $\dfrac{dy}{dx}$ if $2x^2-3xy+4y^2-2x+y+5=0$.

10. If $$x^m y^n=a^{m+n}$$
shew that $$\frac{dy}{dx}=-\frac{my}{nx}.$$

11. If $$\sin^m x\cos^n y=\text{const.}$$
shew that $$\frac{dy}{dx}=\frac{m}{n}\cot x\cot y.$$

CHAPTER XI

DIFFERENTIATION OF n^x, e^x, $\log_e x$. HYPERBOLIC FUNCTIONS. LOGARITHMIC DIFFERENTIATION. COMPOUND INTEREST LAW. SOME DIFFERENTIAL EQUATIONS

245. HERE n is constant and the index x is variable.

With the usual notation put

$$y = n^x,$$

then
$$y + \Delta y = n^{x+\Delta x} = n^x \,.\, n^{\Delta x},$$

$$\therefore \quad \Delta y = n^x (n^{\Delta x} - 1),$$

and
$$\frac{\Delta y}{\Delta x} = n^x \,.\, \frac{n^{\Delta x} - 1}{\Delta x}.$$

Now $\underset{\Delta x \to 0}{\mathrm{Lt}} \dfrac{n^{\Delta x} - 1}{\Delta x}$ is obviously a function of n and is independent of x.

$\therefore$ we have $\dfrac{dy}{dx} = \mathrm{N} \,.\, n^x$ or $\mathrm{N}y$, where N is a constant, i.e. it is the same for all powers of the same number n.

246. We can find an approximate value for this constant in particular cases, e.g. if $n = 2$,

$$\frac{\Delta y}{\Delta x} = 2^x \,.\, \frac{2^{\Delta x} - 1}{\Delta x}.$$

By logarithms we get

$$\frac{2^{\cdot 2} - 1}{\cdot 2} = \cdot 743,$$

$$\frac{2^{\cdot 1} - 1}{\cdot 1} = \cdot 718,$$

$$\frac{2^{\cdot 01} - 1}{\cdot 01} = \cdot 696,$$

$$\frac{2^{\cdot 001} - 1}{\cdot 001} = \cdot 693.$$

The last two were got by using 7-figure tables. With 4-figure tables the best we can do is to say that $\dfrac{2^{\cdot 01}-1}{\cdot 01}=\cdot 7$ approximately. With 7-figure tables it is impossible to distinguish between $\dfrac{2^{\cdot 001}-1}{\cdot 001}$ and $\dfrac{2^{\cdot 0001}-1}{\cdot 0001}$.

$\therefore$ we may say that if $y=2^x$,

$$\frac{dy}{dx}=\cdot 693\times 2^x \text{ approximately.}$$

247. It is important to realise that this number is $\cdot 693$ because we are dealing with powers of 2.

Take the case when $x=5$, $y=2^5$,

if $$x=5\cdot 1,\quad y=2^{5\cdot 1}=2^5\times 2^{\cdot 1},$$

and $$\frac{\text{Increase in } y}{\text{Increase in } x}=\frac{2^{5\cdot 1}-2^5}{\cdot 1}=2^5\times\left[\frac{2^{\cdot 1}-1}{\cdot 1}\right].$$

Similarly if $$x=5\cdot 01,\quad y=2^{5\cdot 01}=2^5\times 2^{\cdot 01},$$

and $$\frac{\text{Increase in } y}{\text{Increase in } x}=2^5\times\left[\frac{2^{\cdot 01}-1}{\cdot 01}\right],$$

and so on.

Take the case when $x=7$, $y=2^7$,

if $$x=7\cdot 1,\quad y=2^{7\cdot 1}=2^7\times 2^{\cdot 1},$$

and $$\frac{\text{Increase in } y}{\text{Increase in } x}=\frac{2^{7\cdot 1}-2^7}{\cdot 1}=2^7\times\left[\frac{2^{\cdot 1}-1}{\cdot 1}\right].$$

Similarly if $$x=7\cdot 01,\quad y=2^{7\cdot 01}=2^7\times 2^{\cdot 01}$$

and $$\frac{\text{Increase in } y}{\text{Increase in } x}=2^7\times\left[\frac{2^{\cdot 01}-1}{\cdot 01}\right],$$

and so on.

The quantities in square brackets are the same whatever power of 2 we start from, and $\cdot 693$ is the limiting value to which the series of fractions $\dfrac{2^{\cdot 1}-1}{\cdot 1}$, $\dfrac{2^{\cdot 01}-1}{\cdot 01}$ etc. continually approach as the index $\rightarrow 0$.

If we are dealing with powers of 3, 1·10 takes the place of ·693 [this being the limiting value to which the series of fractions $\frac{3^{\cdot 1}-1}{\cdot 1}$, $\frac{3^{\cdot 01}-1}{\cdot 01}$ etc. continually approach as the index $\rightarrow 0$.]

And similarly for powers of other numbers.

EXERCISES. LXXV.

As in § 246 shew that

1. If $y=3^x$, $\frac{dy}{dx}=1{\cdot}10\times 3^x$ approximately.

2. If $y=5^x$, $\frac{dy}{dx}=1{\cdot}61\times 5^x$ approximately.

3. If $y=2{\cdot}5^x$, $\frac{dy}{dx}=0{\cdot}92\times 2{\cdot}5^x$ approximately.

4. If $y=2{\cdot}7^x$, $\frac{dy}{dx}=0{\cdot}99\times 2{\cdot}7^x$ approximately.

5. If $y=2{\cdot}8^x$, $\frac{dy}{dx}=1{\cdot}03\times 2{\cdot}8^x$ approximately.

248. The results of Exs. LXXV. 4, 5 suggest that there is a value of n between 2·7 and 2·8 such that if $y=n^x$, $\frac{dy}{dx}=1\times n^x$. This value of n is called e. Its value can be calculated to any degree of accuracy required, by methods which have no place in this book. Correct to 9 decimal places it is 2·718281828.

e is defined to be such that if $y=e^x$, then $\frac{dy}{dx}=y$.

So far as numerical results are concerned, it will be sufficient for us to know that it is 2·7183 correct to 5 significant figures.

EXERCISES. LXXVI.

Shew that approximately $\frac{2{\cdot}7183^{\cdot 001}-1}{\cdot 001}=1$, and hence that if $y=2{\cdot}7183^x$, $\frac{dy}{dx}=2{\cdot}7183^x$ approximately.

249. We can now express the constant N in § 245 in terms of n and e.

First, since
$$\frac{d \,.\, e^x}{dx} = e^x,$$
it follows that
$$\frac{d \,.\, e^{ax}}{dx} = \frac{d \,.\, e^{ax}}{d \,.\, ax} \cdot \frac{d \,.\, ax}{dx} = a \,.\, e^{ax}.$$

Now by the definition of a logarithm
$$n = e^{\log_e n},$$
$$\therefore \quad n^x = e^{x.\log_e n}.$$
$\therefore$ if
$$y = n^x,$$
$$\frac{dy}{dx} = \log_e n \times e^{x \,.\, \log_e n},$$
i.e.
$$\frac{dy}{dx} = \log_e n \times n^x.$$

So that N is $\log_e n$.

250. We have thus two very important results:

if
$$\mathbf{y = e^x, \quad \frac{dy}{dx} = y,}$$
if
$$\mathbf{y = n^x, \quad \frac{dy}{dx} = y \times \log_e n,}$$
where e stands for the number 2·718281828....

251. The results in Exs. LXXV. therefore give us the following approximate values for logarithms to the base e.

No.	log. to base e
2	·693
2·5	·92
2·7	·99
2·8	1·03
3	1·10
5	1·61

252. Tables have been made giving logarithms of numbers to the base e [called natural or Hyperbolic, or Napierian logarithms, the last from the inventor John Napier of Merchiston 1550—1617].

If these are not available, the logarithm of any number to the base e can be obtained from the logarithm to the base 10 by using the formula

$$\log_e n = \frac{\log_{10} n}{\log_{10} e}.$$

This is a particular case of a general theorem, viz.

$$\log_a \mathrm{n} = \frac{\log_b \mathrm{n}}{\log_b \mathrm{a}}.$$

For let $\log_b n = x$ and $\log_b a = y$,

$$\therefore\ n = b^x \ldots(1) \text{ and } a = b^y \ldots(2).$$

To get $\log_a n$ we want n as a power of a.

(2) gives $$b = a^{\frac{1}{y}},$$

$\therefore$ from (1) $$n = \left(a^{\frac{1}{y}}\right)^x = a^{\frac{x}{y}},$$

i.e. $$\log_a n = \frac{x}{y} = \frac{\log_b n}{\log_b a}.$$

The special case when $n = b$ is important. This gives

$$\log_a \mathrm{b} = \frac{1}{\log_b \mathrm{a}}.$$

As we shall be mainly concerned with the bases e and 10 we take the particular results got by substituting e and 10 for a and b.

Thus $$\left.\begin{aligned}\log_e n &= \frac{\log_{10} n}{\log_{10} e}\\ \text{and}\quad \log_e 10 &= \frac{1}{\log_{10} e}\end{aligned}\right\},$$

$$\log_{10} e \text{ [i.e. } \log_{10} 2{\cdot}7183\ldots] \text{ is } {\cdot}43429,$$

and its reciprocal is 2·3026, each being correct to 5 significant figures.

We thus have

$$\log_e n = \log_{10} n \times 2{\cdot}3026 \text{ and } \log_{10} n = \log_e n \times {\cdot}43429,$$

e.g.

$$\log_e 2 = \log_{10} 2 \times 2{\cdot}3026$$
$$= {\cdot}30100 \times 2{\cdot}3026$$
$$= {\cdot}6931.$$

Graphic treatment.

253. Fig. 124 shews the graph of $y = 2^x$.

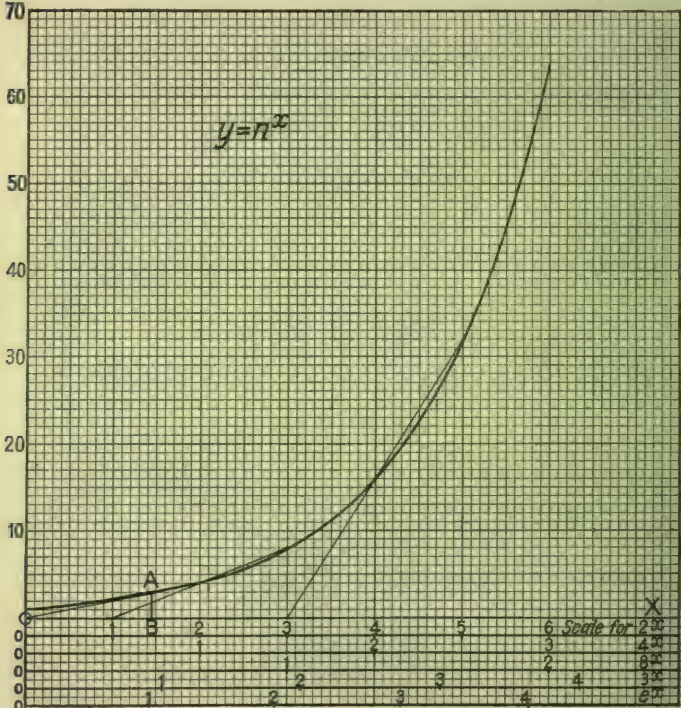


Fig. 124.

Now if we keep the same axes and make a suitable change in the x-scale, this can be made to stand for the graph of $y = n^x$, where n is any number whatever.

e.g. $$4 = 2^2,$$
so that $$4^x = 2^{2x},$$
$$4^3 = 2^6,$$
$$4^{1\cdot 5} = 2^3 \text{ etc.},$$

the index when a number is expressed as a power of 4 being half the index when it is expressed as a power of 2. Therefore if the x-scale be as indicated in the second line, the graph is that of $y = 4^x$.

Similarly since $8 = 2^3$ and $8^x = 2^{3x}$,

the graph will be that of $y = 8^x$ if the x-scale be as indicated in the third line.

Any number can be expressed as a power of 2, for instance

$$3 = 10^{\cdot 4771} \text{ and } 2 = 10^{\cdot 3010},$$

$$\therefore\ 3 = 2^{\frac{\cdot 4771}{\cdot 3010}} = 2^{1\cdot 585}.$$

$\therefore$ if the x-scale be as indicated in the fourth line, the graph will be that of $y = 3^x$.

And so on for any value of n.

The x-axis is in each case OX.

254. We shall now prove an important geometrical property of the curve $y = n^x$ [Fig. 125].

P is the point (a, n^a),

Q is the point $(a + h, n^{a+h})$,

$$\frac{\text{RN}}{\text{PS}} = \frac{\text{NP}}{\text{SQ}},$$

$$\therefore\ \frac{\text{RN}}{h} = \frac{n^a}{n^{a+h} - n^a} = \frac{1}{n^h - 1},$$

$$\therefore\ \text{RN} = \frac{h}{n^h - 1},$$

and is independent of a, i.e. of the position of P on the curve but depends only on the length of the horizontal step PS.

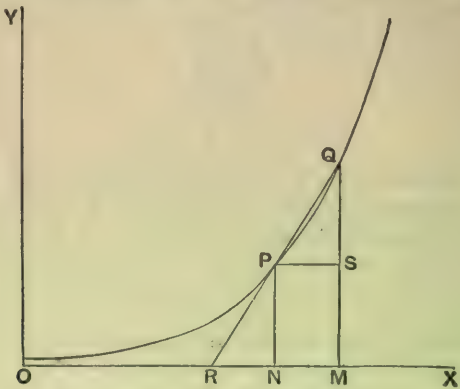


Fig. 125.

EXERCISES. LXXVII.

Verify this property on the graph of $y=2^x$

(i) by joining the points for which $x=2$ and $x=3$ and also those for which $x=4$ and $x=5$.

(ii) by joining the points for which $x=0$ and $x=2$, also those for which $x=2$ and $x=4$, also those for which $x=3$ and $x=5$.

255. Now let $h \to 0$ and we get the property that in the curve $y=n^x$ the subtangent is the same at every point.

i.e. if NP be the ordinate of P and if the tangent at P meet OX in T, then TN is the same for all positions of P on the curve. [Fig. 126.]

Now if P be the point (x, y) and if $\text{TN}=t$, we have

$$y = t \times \text{gradient of TP}$$

$$= t\frac{dy}{dx},$$

$$\therefore \frac{dy}{dx} = \frac{y}{t}, \text{ where } t \text{ is constant.}$$

256. There is one of the curves $y = n^x$ for which $t = 1$, in other words on some scale or other TN will be of unit length.

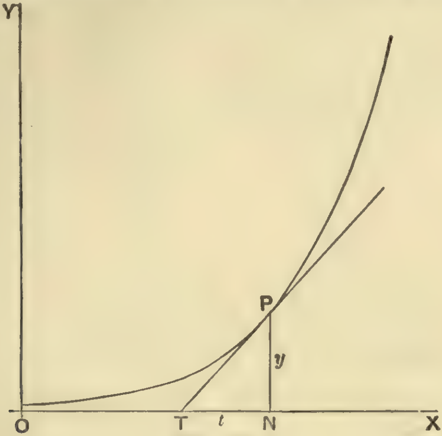


Fig. 126.

Let this particular curve be called $y = e^x$.

Then as before e^x is defined by the property that

$$\text{if } y = e^x, \frac{dy}{dx} = y.$$

257. The numerical value of e can be found approximately from the graph as follows.

Let A be the point on the graph the tangent at which goes through O. [This point can be found fairly accurately with a straight edge.] Then if AB be the ordinate of A and we choose our x-scale so that OB is unit length the graph will be $y = e^x$, and moreover BA gives the value of y corresponding to $x = 1$, i.e. the number of y-units in BA is e.

From Fig. 124 we can see that e is something between 2·5 and 3.

From Fig. 127 which is part of Fig. 124 drawn to a larger scale we see that e is between 2·7 and 2·75.

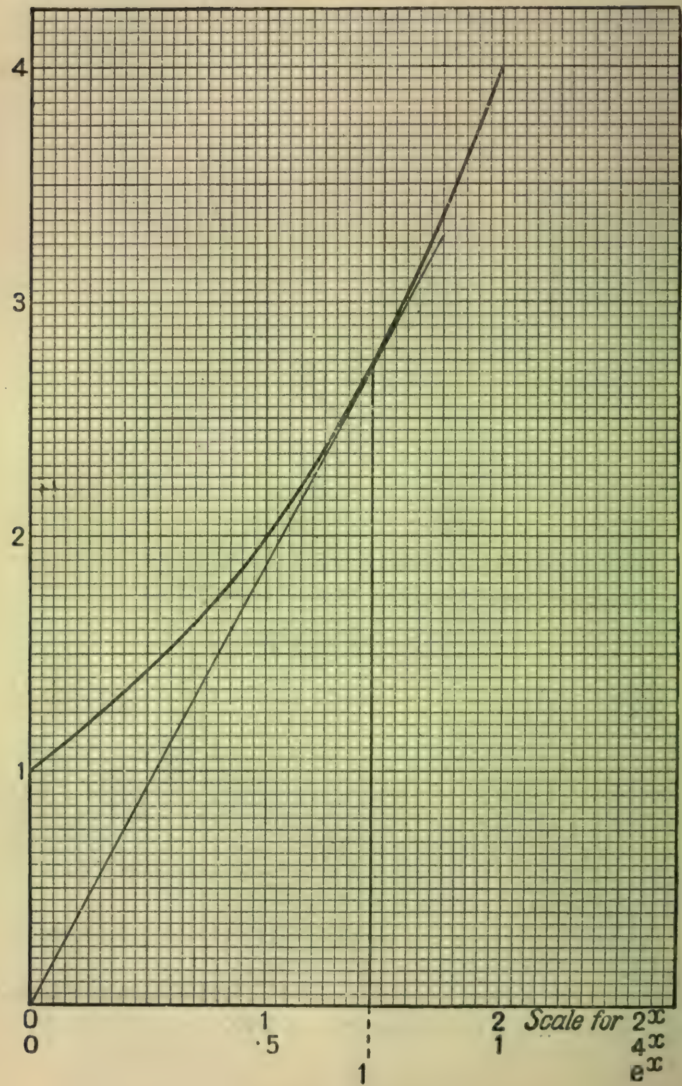


Fig. 127

10. Prove that $y=e^{-\cdot 3x}$ and $y=e^{-\cdot 3x}\sin 2x$ touch each other at the points corresponding to $x=\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$, etc., and find the gradient where $x=\frac{\pi}{4}$.

11. Prove that the curve $y=e^{-\cdot 3x}\sin 2x$ has turning points where

$$x=\frac{a}{2},\ \pm\frac{\pi}{2}+\frac{a}{2},\ \pm\pi+\frac{a}{2},\ \pm\frac{3\pi}{2}+\frac{a}{2},\ \text{etc.,}$$

and points of inflexion where

$$x=a,\ \pm\frac{\pi}{2}+a,\ \pm\pi+a,\ \pm\frac{3\pi}{2}+a,\ \text{etc.,}$$

where $a=\tan^{-1}\frac{20}{3}$.

12. On the same sheet plot between $x=0$ and $x=2\pi$,

(i) $y=e^{-\cdot 3x}$, (ii) $y=\sin 2x$, (iii) $y=e^{-\cdot 3x}\sin 2x$. [Fig. 129.]

Differentiation of $\log_e x$.

258. If
$$y=\log_e x,$$
$$x=e^y,$$
$$\therefore\ \frac{dx}{dy}=x,$$
$$\therefore\ \frac{dy}{dx}=\frac{1}{x}.$$

This is a most important result. We deduce that

$$\int\frac{1}{x}dx=\log_e x+c,$$

$\frac{1}{x}$ being the only power of x we have hitherto been unable to integrate.

259. *Ex.* 1. If $y=\log_{10} x$, we use the fact that

$$\log_{10} x=\log_e x\times\cdot 43429,$$
$$\therefore\ \frac{dy}{dx}=\frac{\cdot 43429}{x}.$$

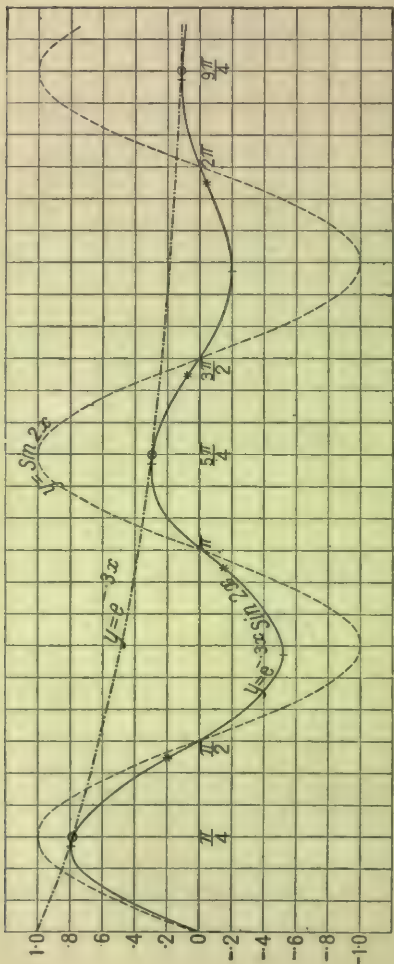


Fig. 129.

Ex. 2. If $$y = \log_e \sin x,$$

$$\frac{dy}{dx} = \frac{d \log_e \sin x}{d \sin x} \cdot \frac{d \sin x}{dx}$$

$$= \frac{1}{\sin x} \cdot \cos x$$

$$= \cot x.$$

EXERCISES. LXXIX.

1. Illustrate the fact that if $y = \log_e x$, $\frac{dy}{dx} = \frac{1}{x}$, by filling up the following table:

x	$y = \log_e x$			
2				
$x + \Delta x$	$y + \Delta y$	Δx	Δy	$\frac{\Delta y}{\Delta x}$
2·1				
2·01				
2·001				

[If you have no table of logs to base e, you must say
$$\log_e 2 = \log_{10} 2 \times 2{\cdot}3026.]$$

Also make a similar table for $x = 6$.

Illustrate also by reference to the graph of $y = \log_e x$. [Fig. 130.]

2. Find $\frac{dy}{dx}$ for the following values of y:

(i) $\log_e 3x$, (ii) $\log_e x^3$, (iii) $\log_e (3x + 5)$,

(iv) $\log_e \cos x$, (v) $\log_e \tan x$, (vi) $\log_e (3x^2 + 5x + 1)$,

(vii) $\log_e \sin 2x$, (viii) $\log_3 x$, (ix) $\log_e \tan \frac{x}{2}$,

(x) $\log_e (\sec x + \tan x)$, (xi) $\log_e \tan \left(\frac{\pi}{4} + \frac{x}{2}\right)$,

(xii) $\log_e \sqrt{x^2 + 1}$, (xiii) $\log_e \{(x + 1)(x + 2)(x + 3)\}$,

(xiv) $\log_e (x + \sqrt{x^2 + a^2})$, (xv) $\log_e (x + \sqrt{x^2 - a^2})$,

(xvi) $\log \frac{a + x}{a - x}$ $(x^2 < a^2)$, (xvii) $\log \frac{x + a}{x - a}$ $(x^2 > a^2)$,

(xviii) $x \log_e x$, (xix) $x^2 \log_e x$, (xx) $\log_e (ax + b)$,

(xxi) $\log_e (ax^2 + bx + c)$, (xxii) $\log_e u$, where u is any function of x.

3. Write down $\int y\,dx$ for the following values of y:

(i) $\frac{1}{x}$, (ii) $\frac{1}{2x}$, (iii) $\frac{1}{2x+3}$,

(iv) $\tan x$, (v) $\operatorname{cosec} x$, (vi) $\sec x$,

(vii) $\frac{2x+3}{x^2+3x+5}$, (viii) $\frac{1}{ax+b}$, (ix) $\frac{x+2}{x^2+4x-7}$,

(x) $\frac{1}{\sqrt{x^2+a^2}}$, (xi) $\frac{1}{\sqrt{x^2-a^2}}$, (xii) $\frac{1}{a^2-x^2}\,(x^2<a^2)$,

(xiii) $\frac{1}{x^2-a^2}\,(x^2>a^2)$, (xiv) $\log_e x$, (xv) $\frac{x^3+2x^2+x}{x^2}$,

(xvi) $\frac{x^2+5x-1}{x+2}$. $\left[\text{Get this equal to } x+3-\frac{7}{x+2}.\right]$

(xvii) $\frac{x^3-x^2-7x+8}{x-3}$.

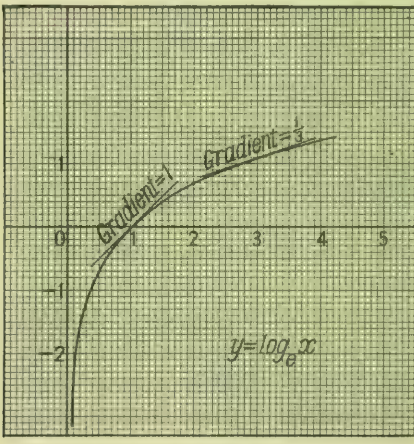


Fig. 130.

4. Shew that $$\int_1^2 \frac{1}{x}\,dx = \int_3^6 \frac{1}{x}\,dx = \int_{50}^{100} \frac{1}{x}\,dx,$$

and generally that $$\int_a^b \frac{1}{x}\,dx = \int_{ka}^{kb} \frac{1}{x}\,dx = \log_e \frac{b}{a}.$$

5. Find the area bounded by

(i) $xy=1,\ y=0,\ x=\frac{1}{2},\ x=3$,

(ii) $xy=1,\ y=0,\ x=3,\ x=18$.

6. Find the work done by a gas in expanding from 2 to 3 cubic feet, the pressure and volume being connected by the law $pv=$const., given that the pressure is 2160 lbs. wt./sq. ft. when the volume is $\frac{1}{4}$ c. ft. [v. Ex. 4, § 184].

7. In the last example find the work required to compress the gas from

(i) 30 cubic feet to 20 cubic feet,

(ii) 3 cubic inches to 2 cubic inches.

8. Evaluate

(i) $\displaystyle\int_0^4 \frac{dx}{\sqrt{x^2+9}}$, (ii) $\displaystyle\int_1^2 \frac{dx}{3x+5}$, (iii) $\displaystyle\int_0^a \frac{dx}{4a^2-x^2}\,(x^2<4a^2)$,

(iv) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x\,dx$, (v) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \operatorname{cosec} x\,dx$.

9. Find the C. G. of the area contained between $xy=1,\ y=0,\ x=1,\ x=4$.

10. Find the C. G. of the area contained between $x^2y=1,\ y=0,\ x=1,\ x=4$.

11. Find the mean value of $\dfrac{1}{x}$ between $x=1$ and $x=2$. [Cp. Exs. XLVI. 3.]

12. P, Q are two points on the same meridian whose latitudes are x, $x+\Delta x$ (radians).

Shew that arc $PQ=k.\Delta x$ nautical miles, where $k=\dfrac{180\times 60}{\pi}$.

On a Mercator chart, taking as unit the distance representing a sea-mile at the equator, the distance corresponding to PQ will be $k.\Delta x$ multiplied by some factor between $\sec x$ and $\sec(x+\Delta x)$, and the whole distance from the equator to a point B, latitude λ, will be represented on the Mercator chart by a distance $\displaystyle\int_0^\lambda k\sec x\,.\,dx$.

Find the distance on a Mercator chart from the equator to a spot in lat. 45° and compare with what you find in the table of meridional parts.

13. For values of $x>-1$

$$2y=x^2-12x+30\log_e(x+2).$$

Find the maximum and minimum values of y, and evaluate them numerically.

260. Another method of approximating to the value of e.

If
$$f(x) = \log_e x,$$
$$f'(x) = \frac{1}{x},$$
and
$$\int_a^b \frac{1}{x}\,dx = \log_e b - \log_e a.$$

Put $b = e$ and $a = 1$ and we have
$$\int_1^e \frac{1}{x}\,dx = \log_e e = 1.$$

Draw the graph of $y = \frac{1}{x}$. Then the area bounded by the curve, the x-axis and the ordinates $x = 1$ and $x = e$ is 1 unit of area, i.e. it is equal to the area of the square OPQR [Fig. 131].

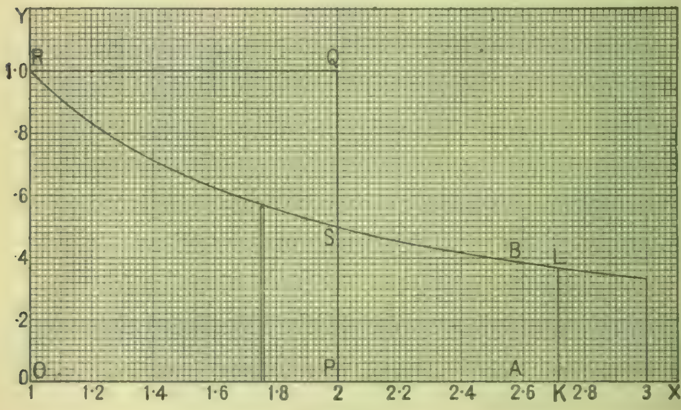


Fig. 131.

If we can place an ordinate KL so that the area OKLR = area OPQR or area PKLS = area SQR the value of x given by OK is e.

Now it is obvious from the figure that e lies between 2·6 and 3 [mere counting of whole square inches is sufficient to settle this].

We shall get a very good approximation to the area OABR by drawing rectangles as in Figs. 68, 69, p. 163, each of width ·01, and taking the arithmetic mean of the sums of inside and outside rectangles. [When x is positive, $\frac{dy}{dx}$ is negative, and $\frac{d^2y}{dx^2}$ positive, so that the curve is like Fig. 40 (c), § 94.]

The sum of the outside rectangles is

$$\cdot 01\left[\frac{1}{1}+\frac{1}{1\cdot 01}+\frac{1}{1\cdot 02}+\ldots\ldots+\frac{1}{2\cdot 59}\right],$$

and the summation is easily effected by means of a table of reciprocals.

Using 4-figure tables [the numbers to be added are very conveniently placed for addition] we get $\cdot 01 \times 95\cdot 8596$.

The sum of the inside rectangles < the sum of the outside rectangles by

$$\cdot 01\left[1-\frac{1}{2\cdot 60}\right] \text{ or } \cdot 01 \times \cdot 6154.$$

$\therefore$ Area OABR < ·958596 and > ·952442,

a good approximation will be **·955519** [as a matter of fact the first 5 figures are right].

Now try OA′B′R where A′B′ is $x = 2\cdot 7$.

Sum of outside rectangles will be greater than in previous case by

$$\cdot 01\left[\frac{1}{2\cdot 60}+\frac{1}{2\cdot 61}+\ldots\ldots+\frac{1}{2\cdot 69}\right], \text{ i.e. by } \cdot 037810,$$

and is therefore **·996406**.

The sum of the inside rectangles is less than this by

$$\cdot 01\left[1-\frac{1}{2\cdot 70}\right] \text{ or } \cdot 006296.$$

$\therefore$ Area OA′B′R < ·996406 and > ·990110,

a good approximation is **·993258**.

Now try $OA''B''R$ where $A''B''$ is $x = 2{\cdot}8$.

$$\text{Area } OA''B''R < 1{\cdot}032839 \text{ and } > 1{\cdot}026410,$$

a good approximation is **1·029624.**

$\therefore$ e is between 2·7 and 2·8.

Try 2·75.

$$\text{Area} < 1{\cdot}014789 \text{ and } > 1{\cdot}008425,$$

a good approximation is **1·011607.**

Try 2·72.

$$\text{Area} < 1{\cdot}003800 \text{ and } > {\cdot}997476,$$

a good approximation is **1·000638**.

Try 2·71.

$$\text{Area} < 1{\cdot}000110 \text{ and } > {\cdot}993800,$$

a good approximation is **·996955.**

$\therefore$ $e < 2{\cdot}72$ and $> 2{\cdot}71$ and apparently nearer to 2·72.

If we use a table of reciprocals which gives more figures and divide the area into strips ·001 x-units wide, we shall be able to shew that

$$e > 2{\cdot}718 \text{ and } < 2{\cdot}719.$$

Hyperbolic Functions.

261. Definitions. $\dfrac{e^x - e^{-x}}{2}$ is called the **hyperbolic sine** of x (written $\mathbf{sinh\,x}$); $\dfrac{e^x + e^{-x}}{2}$ is called the **hyperbolic cosine** of x (written $\mathbf{cosh\,x}$); $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ or $\dfrac{\sinh x}{\cosh x}$ is called the **hyperbolic tangent** of x (written $\mathbf{tanh\,x}$); $\dfrac{1}{\tanh x}$ is $\coth x$, $\dfrac{1}{\cosh x}$ is $\operatorname{sech} x$, and $\dfrac{1}{\sinh x}$ is $\operatorname{cosech} x$.

EXERCISES. LXXX.

1. Prove $\cosh^2 x - \sinh^2 x = 1$.

[Just as the co-ordinates of a point on the circle $x^2+y^2=a^2$ may be taken to be $(a\cos\theta,\ a\sin\theta)$, so the co-ordinates of any point on the rectangular hyperbola $x^2-y^2=a^2$ may be taken to be $(a\cosh\theta,\ a\sinh\theta)$.]

2. Shew that **cosh x is never less than 1**, by writing it

$$\frac{(e^{\frac{x}{2}}-e^{-\frac{x}{2}})^2}{2}+1.$$

3. Shew that $\sinh(-x) = -\sinh x$ and $\cosh(-x) = \cosh x$.

4. On the same sheet draw (i) $y=e^x$, (ii) $y=e^{-x}$, (iii) $y=\sinh x$, (iv) $y=\cosh x$ between $x=-2{\cdot}5$ and $x=2{\cdot}5$.

5. Shew that **tanh x lies between − 1 and + 1.**

$\left[\text{Write it } 1-\dfrac{2}{e^{2x}+1} \text{ or } -1+\dfrac{2}{e^{-2x}+1}.\right]$

Shew also that $\tanh(-x) = -\tanh x$ and draw the graph of $y=\tanh x$ between $x=-2{\cdot}5$ and $x=2{\cdot}5$.

6. Shew that

$$\begin{aligned}\cosh 2x &= \cosh^2 x + \sinh^2 x\\ &= 2\cosh^2 x - 1\\ &= 1 + 2\sinh^2 x\end{aligned}$$

and

$$\sinh 2x = 2\sinh x\cosh x.$$

7. Shew that

(i) $\dfrac{d\sinh x}{dx} = \cosh x$, (ii) $\dfrac{d\cosh x}{dx} = \sinh x$,

(iii) $\dfrac{d\tanh x}{dx} = \operatorname{sech}^2 x = 1-\tanh^2 x$.

8. If $y = A\cosh x + B\sinh x$, shew that $\dfrac{d^2y}{dx^2} - n^2y = 0$.

9. What are

(i) $\displaystyle\int \sinh x\,dx$, (ii) $\displaystyle\int \cosh x\,dx$?

10. Shew that

$$\int \sinh^2 x\,dx = -\frac{x}{2}+\frac{\sinh 2x}{4},$$

$$\int \cosh^2 x\,dx = \frac{x}{2}+\frac{\sinh 2x}{4}. \quad [\text{v. Ex. 6.}]$$

11. **If $y=\sinh^{-1}x$** [i.e. $x=\sinh y$], prove $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{1+x^2}}$;

and **if $y=\sinh^{-1}\dfrac{x}{a}$**, $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{a^2+x^2}}$.

[Remember $\cosh y$ is necessarily positive.]

12. If $y=\sinh^{-1}x$, prove $e^y=x+\sqrt{x^2+1}$, i.e. $\sinh^{-1}x$ is the same as $\log_e(x+\sqrt{x^2+1})$. Similarly shew that $\sinh^{-1}\dfrac{x}{a}$ is the same as

$$\log_e\frac{x+\sqrt{x^2+a^2}}{a}.$$

[Compare results of Exs. LXXIX. 2 (xiv) and Exs. LXXX. 11.]

13. **If $y=\cosh^{-1}x$** $(x>1)$, prove $\dfrac{dy}{dx}=\pm\dfrac{1}{\sqrt{x^2-1}}$;

and **if $y=\cosh^{-1}\dfrac{x}{a}$**, prove $\dfrac{dy}{dx}=\pm\dfrac{1}{\sqrt{x^2-a^2}}$.

14. If $y=\cosh^{-1}x$, prove $e^y=x\pm\sqrt{x^2-1}$, i.e. $\cosh^{-1}x$ is the same as $\log_e(x\pm\sqrt{x^2-1})$ or $\pm\log(x+\sqrt{x^2-1})$, since

$$x-\sqrt{x^2-1}=\frac{1}{x+\sqrt{x^2-1}}.$$

Similarly shew that $\cosh^{-1}\dfrac{x}{a}$ is the same as $\log\dfrac{x\pm\sqrt{x^2-a^2}}{a}$.

[Compare results of Exs. LXXIX. 2 (xv) and Exs. LXXX. 13.]

15. **If $y=\tanh^{-1}x$** $(x^2<1)$, prove $\dfrac{dy}{dx}=\dfrac{1}{1-x^2}$;

and **if $y=\tanh^{-1}\dfrac{x}{a}$** $(x^2<a^2)$, prove $\dfrac{dy}{dx}=\dfrac{a}{a^2-x^2}$.

16. **If $y=\coth^{-1}x$** or $\tanh^{-1}\dfrac{1}{x}$ $(x^2>1)$, prove $\dfrac{dy}{dx}=-\dfrac{1}{x^2-1}$;

and **if $y=\coth^{-1}\dfrac{x}{a}$** or $\tanh^{-1}\dfrac{a}{x}$ $(x^2>a^2)$, prove $\dfrac{dy}{dx}=-\dfrac{a}{x^2-a^2}$.

17. If $y=\tanh^{-1}x$ $(x^2<1)$, prove $e^{2y}=\dfrac{1+x}{1-x}$, i.e. $\tanh^{-1}x$ is the same as $\dfrac{1}{2}\log_e\dfrac{1+x}{1-x}$. Similarly $\tanh^{-1}\dfrac{x}{a}$ $(x^2<a^2)$ is the same as $\dfrac{1}{2}\log\dfrac{a+x}{a-x}$.

Also if $y=\coth^{-1}\dfrac{x}{a}$ $(x^2>a^2)$, prove $y=\dfrac{1}{2}\log\dfrac{x+a}{x-a}$.

[Compare results of Exs. LXXIX. 2 (xvi and xvii) and Exs. LXXX. 15, 16.]

262. Note. If $y=\sinh^{-1}x$, the quadratic for e^y gives

$$e^y = x \pm \sqrt{x^2+1},$$

but since e^y is essentially positive, e^y must be $x+\sqrt{x^2+1}$ and there is no ambiguity.

If $y=\cosh^{-1}x$, $e^y = x \pm \sqrt{x^2-1}$ and both values are admissible, i.e. for any value of x (>1) there are two equal and opposite values of y such that $\cosh y = x$. If we take the positive value, $\frac{dy}{dx}$ will be $\frac{1}{\sqrt{x^2-1}}$; if we take the negative value, $\frac{dy}{dx}$ will be $-\frac{1}{\sqrt{x^2-1}}$. These facts will be readily seen from the graphs of $y=\cosh^{-1}x$ and $y=\sinh^{-1}x$.

If we agree that $\cosh^{-1}x$ shall mean the positive value, i.e. if we take $\cosh^{-1}x$ as being $\log_e(x+\sqrt{x^2-1})$ we may say that $\frac{\mathbf{d}\cosh^{-1}\mathbf{x}}{\mathbf{dx}} = \frac{\mathbf{1}}{\sqrt{\mathbf{x}^2-\mathbf{1}}}$.

If $y=\tanh^{-1}x\,(x^2<1)$, $e^{2y}=\frac{1+x}{1-x}$ and $e^y = \pm\sqrt{\frac{1+x}{1-x}}$, but as before, e^y must be positive, so that

$$e^y = \sqrt{\frac{1+x}{1-x}} \text{ and } y = \log\sqrt{\frac{1+x}{1-x}}.$$

In this case therefore there is no ambiguity.

Logarithmic Differentiation.

263. The process of differentiating a product is often simplified by taking logarithms before differentiating.

Ex. 1. Find $\frac{dy}{dx}$ if $y=\frac{(x+3)^2(2x-1)^3}{(3x+2)^5}$.

We have

$$\log y = 2\log(x+3) + 3\log(2x-1) - 5\log(3x+2),$$

$$\therefore \frac{1}{y}\cdot\frac{dy}{dx} = \frac{2}{x+3} + \frac{6}{2x-1} - \frac{15}{3x+2}.$$

$$\therefore \quad \frac{dy}{dx} = y\left(\frac{2}{x+3} + \frac{6}{2x-1} - \frac{15}{3x+2}\right)$$
$$= -\frac{(7x-77)(x+3)(2x-1)^2}{(3x+2)^6}.$$

Ex. 2. If $y = uvwz$ where u, v, w, z are functions of x,

$$\log y = \log u + \log v + \log w + \log z.$$

$$\therefore \quad \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} + \frac{1}{v} \cdot \frac{dv}{dx} + \frac{1}{w} \cdot \frac{dw}{dx} + \frac{1}{z} \cdot \frac{dz}{dx}.$$

$$\therefore \quad \frac{dy}{dx} = vwz \cdot \frac{du}{dx} + wzu \cdot \frac{dv}{dx} + zuv \cdot \frac{dw}{dx} + uvw \cdot \frac{dz}{dx}.$$

(Cp. § 219.)

EXERCISES. LXXXI.

Find $\frac{dy}{dx}$ if

1. $y = \frac{(3x+1)^7 (4x-3)^5}{(5x+2)^4}$.

2. $y = \frac{(x^3+1)^2 \sqrt{x+2}}{\sqrt[3]{3x+1}}$.

3. $y = \sqrt{\frac{(3x^3+1)(2x-3)^3}{4x^2+2x+1}}$.

4. $y = \frac{e^{\sin x} \cdot \cos^3 x}{(4x+1)}$.

5. $y = x^x$.

6. $y = x^{\frac{1}{x}}$.

7. $y = (\sin x)^{\cos x}$.

The Compound Interest Law.

264. Suppose money put out to compound interest at 5 %, and suppose in the first instance that interest is payable yearly.

Then if £x be the capital at the beginning of any year, the increase in capital during the year is £$\cdot 05x$.

If interest be payable half-yearly, the increase in capital during any half-year is $\cdot 05x \times \frac{1}{2}$ where £x is the capital at the beginning of the half-year.

If interest be payable every $\frac{1}{n}$th of a year, the increase in capital during any period is $\cdot 05x \times \frac{1}{n}$ where £x is the capital at the beginning of the period.

Now suppose the increase in capital to be continuous and not to increase suddenly at the end of stated periods.

If £x be the capital at the end of t years, the increase (Δx) during the next interval Δt is between

$$\cdot 05x \,.\, \Delta t \quad\text{and}\quad \cdot 05\,(x + \Delta x)\,.\,\Delta t,$$

$$\therefore\quad \frac{\Delta x}{\Delta t} \text{ is between } \cdot 05x \text{ and } \cdot 05\,(x + \Delta x)\,;$$

$$\therefore\quad \frac{dx}{dt} = \cdot 05x,$$

and generally if interest is at $r\,\%$

$$\frac{dx}{dt} = \frac{rx}{100} = kx \text{(where £}k = \text{Int. on £1 for 1 yr. at } r\,\%),$$

i.e. x is a function of t such that $\frac{dx}{dt}$ is proportional to x.

i.e. x is a function of the form Ae^{kt}.

Now when $t = 0$, $x = P$ [£P being original capital].

$$\therefore\quad P = Ae^0 = A.$$

$$\therefore\quad x = Pe^{kt} = Pe^{\frac{rt}{100}}.$$

265. If y is a function of x such that

$$\left.\begin{array}{c} y = Ae^{kx} \\ \text{or} \qquad \dfrac{dy}{dx} \propto y \end{array}\right\},$$

y and x are said to be connected by a Compound Interest Law.

Notice if x increases in A.P., y increases in G.P., for suppose x takes successively values

$$a, \quad a + b, \quad a + 2b, \quad \ldots,$$

the corresponding values of y are

$$Ae^{ka}, \quad Ae^{k(a+b)}, \quad Ae^{k(a+2b)}, \quad \ldots$$

and these form a G.P. with common ratio e^{kb}.

Notice also that A is the value of y when x = 0.

Examples of the Compound Interest Law.

266. (1) **Newton's Law of Cooling.** Under certain conditions the rate of fall of temperature of a cooling body is proportional to the excess of its temperature above that of surrounding bodies.

i.e. if $\theta°$ be this excess of temperature at the end of t seconds,

$$\frac{d\theta}{dt} = -k\theta, \left[\begin{array}{c}\text{the temperature of surrounding bodies} \\ \text{being supposed to remain constant}\end{array}\right]$$

where k is some constant. [The sign is $-$ to indicate that θ decreases as t increases.]

From this we get $\qquad \theta = Ae^{-kt},$

where A is some constant.

The constants A and k can be determined experimentally by making two observations.

e.g. Suppose when $\quad t = 10, \quad \theta = 17$
and when $\quad t = 30, \quad \theta = 12$,

then

$$\left.\begin{array}{l} 17 = Ae^{-10k} \\ 12 = Ae^{-30k} \end{array}\right\},$$

$$\therefore \quad A^2 = \frac{17^3}{12}.$$

$$\therefore \quad A = 20{\cdot}2.$$

And

$$\frac{17}{12} = e^{20k},$$

$$\therefore \quad k = \frac{1}{20}\log_e \frac{17}{12}$$

$$= {\cdot}017.$$

$$\therefore \quad \theta = 20{\cdot}2e^{-{\cdot}017t}.$$

(2) **Atmospheric pressure.** Let the pressure be p lbs. per sq. ft. at a height h ft. above the earth's surface, and let $(p + \Delta p)$ be the pressure at height $h + \Delta h$ [Δp is negative if Δh is positive].

Also let w lbs. be the weight of a cubic foot of air at pressure p and $(w + \Delta w)$ its weight at pressure $(p + \Delta p)$.

Then $-\Delta p$ is the weight of Δh cubic feet of air whose average density is between w and $w + \Delta w$.

i.e. $-\Delta p$ is between $w\Delta h$ and $(w + \Delta w)\,\Delta h$,

or $$\frac{dp}{dh} = -w.$$

Now if the temperature be supposed constant w varies as p by Boyle's Law, or $w = w_0 \dfrac{p}{p_0}$, where p_0 lbs./sq. ft. is the pressure and w_0 lbs./c. ft. the density at the surface.

$$\therefore\ \frac{dp}{dh} = -\frac{w_0}{p_0}p.$$

$$\therefore\ p = Ae^{-\frac{w_0}{p_0}h},$$

and when $h = 0$, $p = p_0$, $\therefore\ A = p_0$.

$$\therefore\ p = p_0 e^{-\frac{w_0}{p_0}h}.$$

e.g. if the pressure at the earth's surface be 2100 lbs./sq. foot and the density of air at the surface be ·08 lbs./cubic foot, and if the temperature be supposed the same from the surface to height h,

$$p = 2100e^{-\cdot 00004h}.$$

(3) A body is moving in a straight line and the retardation at any instant $= kv$ ft./sec.^{2}, where k is constant and v ft./sec. the speed at that instant.

If v_0 ft./sec. is the speed when $t = 0$ find formulae for

(i) the speed at the end of t secs.,

(ii) the distance travelled in t secs.

We have
$$\frac{dv}{dt} = -kv.$$
$$\therefore\ v = Ae^{-kt},$$
where A is constant; and $v = v_0$ when $t = 0$, $\therefore\ A = v_0$
$$\therefore\ \mathbf{v = v_0 e^{-kt}}, \quad \text{(i)}$$
i.e.
$$\frac{ds}{dt} = v_0 e^{-kt}.$$
$$\therefore\ s = -\frac{v_0}{k} e^{-kt} + c,$$
and $s = 0$ when $t = 0$, $\quad \therefore\ c = \frac{v_0}{k}$.
$$\therefore\ \mathbf{s = \frac{v_0}{k}(1 - e^{-kt})}.$$

267. We have found in § 264 that the amount at the end of t years of £P at Compound Interest, the increase in capital being supposed continuous, is
$$Pe^{kt} \text{ where } k = \frac{r}{100}.$$

This was obtained by consideration of the increase of capital in a small time Δt, which led to the differential equation
$$\frac{dx}{dt} = kx.$$

We shall now consider the question from a different point of view.

Suppose interest paid at intervals of $\frac{1}{n}$th of a year. Then if £x be the capital at the beginning of any interval, the capital at the end will be $x\left(1 + \frac{k}{n}\right)$.

Thus we have Original capital $= \text{P}$.

$\therefore$ Capital at end of first interval $= \text{P}\left(1+\frac{k}{n}\right)$.

$$\text{,,} \quad \text{,,} \quad \text{2nd} \quad \text{,,} \quad = \text{P}\left(1+\frac{k}{n}\right)\left(1+\frac{k}{n}\right)$$

$$= \text{P}\left(1+\frac{k}{n}\right)^2.$$

$$\text{,,} \quad \text{,,} \quad \text{3rd} \quad \text{,,} \quad = \text{P}\left(1+\frac{k}{n}\right)^3.$$

etc.

$$\text{,,} \quad \text{,,} \quad t \text{ years} \quad = \text{P}\left(1+\frac{k}{n}\right)^{nt}.$$

The amount when the increase in capital is supposed continuous is the limit of this when x is increased indefinitely.

$\therefore$ we have $\text{Lt}\left(1+\frac{k}{n}\right)^{nt}$ when n is increased indefinitely is e^{kt}.

Special cases.

$k=1$, $\qquad \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^{nt}=e^t$,

$k=1$ and $t=1$, $\qquad \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^{n}=e$.

To illustrate the meaning of this last result calculate by logarithms

$$\left(1+\frac{1}{1}\right)^1,\ \left(1+\frac{1}{2}\right)^2,\ \left(1+\frac{1}{3}\right)^3,\ \left(1+\frac{1}{4}\right)^4,\ \left(1+\frac{1}{10}\right)^{10},$$
$$\left(1+\frac{1}{50}\right)^{50},\ \left(1+\frac{1}{100}\right)^{100},\ \left(1+\frac{1}{1000}\right)^{1000}.$$

You should get

2, 2·25, 2·37, 2·44, 2·594, 2·692, 2·705, 2·717.

The theorem is that as n is increased we get nearer and nearer to e (2·7182818...) and can get as near to e as we like by taking n large enough.

EXERCISES. LXXXII.

1. A body starts 2 feet from O and moves so that when it is x feet from O its speed is $3x$ feet/second.

Find a formula giving its distance from O at the end of t seconds.

Find the distance travelled in 3 seconds and the time taken to travel 100 feet.

2. A rope passes round a drum, centre O, A and B being the points where the rope leaves the drum.

P is a point on the drum such that $\angle \mathrm{AOP} = \theta$ radians.

The tension of the rope varies according to the law

$$\frac{d\mathrm{T}}{d\theta} = \mu \mathrm{T} \text{ where } \mu \text{ is the coefficient of friction.}$$

If T_0 is the tension at A and T_1 at B, shew that $\mathrm{T}_1 = \mathrm{T}_0 e^{\mu\alpha}$ where α is the angle AOB.

If $\mu = \cdot 7$, $\mathrm{T}_0 = 20$ (lbs. wt.), arc AB $= 1$ foot and diameter of drum $= 1$ foot, find T_1. Also find at what point the tension is 40 lbs. wt.

3. A cistern full of water has a leak in the bottom. Assuming that the rate at which the water escapes is proportional to the pressure, shew that the rate at which the height of water in the cistern diminishes is proportional to the height

$$\left[\text{i.e. } \frac{dx}{dt} = -kx\right].$$

If the height of the cistern is 10 feet, and the level falls 1 foot in the first minute, when will the cistern be half empty?

4. Suppose a fly-wheel, moment of inertia I lbs. ft.^{2}, rotating in a fluid which produces a resisting torque $c\omega$ lb. ft., where ω is the angular velocity; shew that the angular velocity at the end of time t is given by

$$\omega = \omega_0 e^{kt},$$

where k is a constant, and ω_0 is the initial angular velocity, and find k in terms of c and I.

If the wheel is initially rotating at 200 revolutions a minute, and after a minute at 100 revolutions a minute, after what time will the number of revolutions per minute be (i) 50, (ii) 25, (iii) 20?

5. Mallock's formula for the retardation due to air resistance of a projectile is $k(v-850)$, where v ft./sec. is the speed and k a constant depending on the form and weight of the shell. If V stands for $(v-850)$, shew that the acceleration is $\dfrac{dV}{dt}$.

Hence get a formula giving the speed at the end of t seconds.

6. A body is moving in a straight line and the retardation at any instant is $0{\cdot}03v$ ft./sec.^{2}, where v ft./sec. is the speed at that instant. If the initial speed is 1·5 ft./sec., find the distance of the body at the end of 2 minutes from its position when $t=0$ and shew that it can never reach a point 50 ft. from that position.

7 A rod hanging vertically carries a weight of 100 lbs. The weight of the rod is 1·5 oz. per cubic inch. Find the law connecting the cross-section with the distance from the end if the tensile stress is everywhere 400 lbs./sq. in. (Fig. 132.) [y sq. ins. is area of cross-section x ins. from lower end.]

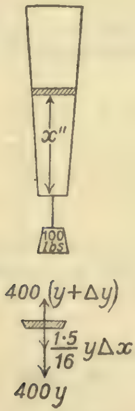


Fig. 132.

EXERCISES. LXXXIII.

1. If $y = \mathrm{A}\cos nx + \mathrm{B}\sin nx$, where A and B are any constants whatever, shew that
$$\frac{d^2y}{dx^2} + n^2y = 0.$$

2. If $y = \mathrm{A}e^{nx} + \mathrm{B}e^{-nx}$ or if $y = \mathrm{A}\cosh nx + \mathrm{B}\sinh nx$, shew that
$$\frac{d^2y}{dx^2} - n^2y = 0.$$

3. Write down solutions (each involving two arbitrary constants) of
$$\text{(i)}\quad \frac{d^2y}{dx^2} + 4y = 0, \qquad \text{(ii)}\quad \frac{d^2y}{dx^2} - 4y = 0.$$

4. If $\dfrac{d^2y}{dx^2} + 4y = 0$ and if, when $x = 0$, $y = 5$ and $\dfrac{dy}{dx} = 4$, find y in terms of x.

5. If $\dfrac{d^2y}{dx^2} - 4y = 0$ and if, when $x = 0$, $y = 5$ and $\dfrac{dy}{dx} = 4$, find y in terms of x.

6. A body moves in a straight line and s ft. is its distance from a fixed point O in the line at the end of t seconds. The acceleration is $9s$ ft./sec.^{2} towards O.

When $t = 0$, the body is 15 ft. to the right of O and when $t = \dfrac{\pi}{6}$, it is 20 ft. to the right of O.

Shew that
$$s = 15\cos 3t + 20\sin 3t$$
$$= 25\sin(3t + \theta), \text{ where } \theta = \tan^{-1}\frac{3}{4}.$$

Hence shew that the body oscillates backwards and forwards between two points 25 feet on either side of O, and that the time of a complete oscillation is $\dfrac{2\pi}{3}$ seconds.

What is the speed of the body

(i) when $t = 0$, (ii) when it passes through O?

7. If $y = ze^{-2x}$, where z is a function of x, shew that
$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = e^{-2x}\left(\frac{d^2z}{dx^2} + 4z\right)$$
and hence that if $\quad z = \mathrm{A}\cos 2x + \mathrm{B}\sin 2x,$

or $\quad y = e^{-2x}(\mathrm{A}\cos 2x + \mathrm{B}\sin 2x),$
$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 0.$$

8. If $y = ze^{\frac{5}{2}x}$, where z is a function of x, shew that

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{\frac{5}{2}x}\left(\frac{d^2z}{dx^2} - \frac{1}{4}z\right),$$

and hence that if $z = \mathsf{A}e^{\frac{1}{2}x} + \mathsf{B}e^{-\frac{1}{2}x}$ or $y = \mathsf{A}e^{3x} + \mathsf{B}e^{2x}$,

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$$

9. If $y = ze^{-3x}$, where z is a function of x, shew that

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{-3x}\frac{d^2z}{dx^2},$$

and hence that if $z = \mathsf{A}x + \mathsf{B}$ or $y = e^{-3x}(\mathsf{A}x + \mathsf{B})$,

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0.$$

10. If $y = ze^{-px}$, where z is a function of x and p, q are constants, shew that

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + qy = e^{-px}\left[\frac{d^2z}{dx^2} + (q - p^2)z\right].$$

If $\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + qy = 0$, what value should z have

(i) if $q > p^2$, (ii) if $q < p^2$, (iii) if $q = p^2$?

11. By the device indicated in Exs. 7—10 get solutions of

(i) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$, (ii) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$,

(iii) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$, (iv) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0$,

(v) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 7y = 0$, (vi) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + \frac{25}{4}y = 0$.

12. Find A so that if $y = \mathsf{A}e^{2x}$, $\frac{d^2y}{dx^2} - 5y = 3e^{2x}$.

13. Find A so that if $y = \mathsf{A}\sin 2x$, $\frac{d^2y}{dx^2} + 5y = 3\sin 2x$.

14. Find A and B so that if $y = \mathsf{A}\sin 6x + \mathsf{B}\cos 6x$,

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 5\sin 6x.$$

15. If $y = xe^{3x}$, shew that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x}$.

16. Find A so that if $y = \mathrm{A}xe^{2x}$, $\frac{d^2y}{dx^2} - 4y = 3e^{2x}$.

17. Shew that $\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + qy = r$, where p, q, r are constants, is the same as $\frac{d^2u}{dx^2} + 2p\frac{du}{dx} + qu = 0$, where u is $\left(y - \frac{r}{q}\right)$; and hence that the solutions of $\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + qy = r$ are got by adding the constant quantity $\frac{r}{q}$ to those of

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + qy = 0.$$

18. Write down solutions of

(i) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 6$,

(ii) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 7y = 11$,

(iii) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + \frac{25}{4}y = 5$. (See Ex. 11.)

MISCELLANEOUS EXAMPLES ON CHAPTERS IX–XI.

A.

1. A ladder 20 feet long has one end on the ground and the other in contact with a vertical wall. The lower end slips along the ground. Shew that when the foot of the ladder is 16 feet away from the wall the upper end is moving $1\frac{1}{3}$ times as fast as the foot.

2. Evaluate
$$\int_0^{\frac{\pi}{4}}(7\cos\theta - 2\sin\theta + 1)\,d\theta,$$
$$\int_1^2\left(x + \frac{1}{x}\right)dx.$$

3. A point moves in a straight line according to the law $\frac{ds}{dt} = 20\cos 3t$. Prove that the distance described from $t=0$ until it comes to rest is $6\frac{2}{3}$ feet.

4. A variable torque produced by a couple of moment M $\sin\theta$ lb. ft. acts on a shaft, M being constant and θ the angle turned through by the shaft. Find the work done in turning the shaft from $\theta=0$ to $\theta=\pi$.

5. Find the mean value of $\tan x$ between
$$x = \frac{\pi}{6} \text{ and } x = \frac{2\pi}{9}.$$
Check your result by finding the Arithmetic mean of
$$\tan 30°, \quad \tan 31°, \dots \tan 40°.$$

B.

1. Find $\frac{dy}{dx}$ in the following cases:

(i) $y = \sin^2 3x$, (ii) $y = x^{-\frac{1}{2}}\cos 4x$, (iii) $y = \log_e \sec\frac{x}{2}$,

(iv) $y = e^{-3x}\sin\left(x + \frac{\pi}{6}\right)$, (v) $x^2 - 2xy - 3y^2 = 0$.

2. Write down the values of

$$\text{(i)} \int e^{-3x}\,dx, \qquad \text{(ii)} \int (2\sin 3x + 4\cos 6x)\,dx,$$

$$\text{(iii)} \int_0^{\frac{\pi}{4}} \frac{dx}{\cos^2 x}.$$

3. A rod 10 feet long slides with its ends A, B on two lines at right angles which intersect in O.

If A has a uniform speed of 3 ft./sec., find the speed and acceleration of B when A is 8 ft. from O.

4. If $y = xe^{-mx}$ (m positive), find the maximum value of y.

5. O is the centre of a circle, radius $2a$. From a point P inside the circle, PB and PC are drawn to the circumference each of length a. If $\mathrm{OP} = x$, shew that for the figure PBC (bounded by PB, PC and arc BC) to have maximum area

$$x^4 - 9a^2x^2 + 12a^4 = 0.$$

C.

1. If
$$y = a \log \frac{a + \sqrt{a^2 - x^2}}{x} - \sqrt{a^2 - x^2},$$

find $\dfrac{dy}{dx}$ and find the length of the tangent intercepted between the curve and the y-axis for the point where $x = h$.

2. Find the area bounded by the curve

$$y = \sin^2 \frac{\pi x}{2}, \quad y = 0, \quad x = 0, \quad x = 1,$$

(i) by integration, (ii) by Simpson's rule (11 ordinates). Draw a figure.

3. The height of a tower is calculated from the observation of the elevation θ at a distance d ft.

If $d = 200$ and $\theta = 35°$, find the consequent error in the calculated height due to (i) an error of 1 inch in measuring d, (ii) an error of $10'$ in measuring θ.

4. A chord divides a circle into two segments of heights h_1, h_2. If A is the area of either segment and θ the angle which its arc subtends at the centre, prove

$$\frac{dA}{d\theta} = h_1 h_2.$$

5. In a simple horizontal engine a is the length of the crank and l the length of the connecting rod.

Shew that when the crank makes an angle θ with the line of dead centres the angular velocity of the connecting rod is

$$\frac{a\cos\theta}{\sqrt{l^2 - a^2\sin^2\theta}} \cdot \omega,$$

where ω is the angular velocity of the crank.

D.

1. (i) Find $\frac{dy}{dx}$ when $y = \log\frac{x^2+x+1}{x^2-x+1}$.

(ii) Find $\frac{d\sqrt{\frac{1+x}{1-x}}}{dx^2}$.

2. Find the intersections of

$$y = \frac{x}{x^2-2} \text{ with } y = 1,$$

and find the equations of the normals at these points.

3. Sketch roughly the curve $y = 3\cos x - 2\sin x$ between $x = 0$ and $x = \frac{\pi}{2}$.

Find the area bounded by the axes and that part of the curve which lies between $x = 0$ and the point where the curve first cuts the x-axis.

4. The blade of a fan consists of a uniform circular disc, centre O, from which a small portion has been cut away by a chord AB equal to the radius r. Find the distance of the C.G. from AB.

5. The thickness x at the distance r from the centre of a disc of varying thickness is given by $x = be^{-ar^2}$. Find the volume of the disc, if its radius is R. What is it in the special case when $R = 4$, $b = \cdot 5$, $a = \cdot 08$?

E.

1. Find $f'(x)$ and $f''(x)$ for the following values of $f(x)$:

(i) $x\log_e x$, (ii) $e^x \sin x$, (iii) $\frac{\sin x}{x}$, (iv) $\sin^4 x$.

2. Draw a triangle BAC, the angle A being about 120°, and produce AC to D. Let AD represent the plan of a door hinged at A, which is being shut into a position along BA produced by a rod BC turned round the fixed point B by a spring and sliding on the door at C.

If the angular velocity of the door is ω, find the velocity with which the end C of the rod is sliding along the door at any instant.

3. If $$x = a\cos^3\theta, \qquad y = a\sin^3\theta,$$
express $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ as functions of θ.

4. c is the distance between the centres of two spheres, radii a and b. Find a point in the line joining the centres so that the total spherical surface visible from it shall be a maximum.

[Surface of segment of sphere $= 2\pi rh$, where r is radius of sphere and h height of segment.]

5. The half water line section of a small boat is bounded by the curve
$$x = 2 + 2t - 2\cos\frac{\pi t}{10},$$
$$y = 2\sin\frac{\pi t}{10},$$
t ranging from 0 to 10.

Determine the area of the water line section.

F.

1. Differentiate with respect to x:
$$\frac{3-5x}{x^2-x+1}, \qquad \cos^3 2x, \qquad e^{ax}\sin bx, \qquad \log_e\tan\left(2x+\frac{\pi}{4}\right).$$
If $$s = be^{nt} + ce^{-nt},$$
where b, c, n are constants, shew that the acceleration is proportional to the displacement.

2. In the catenary
$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) \text{ or } y = c\cosh\frac{x}{c};$$
shew that the length of the perpendicular let fall from N, the foot of the ordinate PN, upon the tangent at P is of constant length.

Also if the normal at P meet the axis of x in G, shew that PG varies as PN^2.

3. A stick 3 feet long rotates about one end in a vertical plane with uniform angular velocity, making one complete revolution in each second. A light is 6 feet vertically above the centre of rotation and casts a shadow of the stick on the floor which is 6 feet below the centre. Find the speed of the end of the shadow when the stick makes an angle of (i) 30°, (ii) 60°, (iii) 90° with the downward vertical.

4. Shew that the radius of gyration of a triangular lamina about an axis through its C.G. perpendicular to its plane is

$$\frac{\sqrt{a^2+b^2+c^2}}{6}.$$

5. A torque acts on a shaft. When the shaft has turned through an angle θ the torque is $G \sin\theta \sin(\theta - a)$ lb.-ft., where G and a are constant.

Find the work done in a complete revolution.

G.

1. Find $\dfrac{dy}{dx}$ for the following values of y:

(i) $\dfrac{2+x^2}{1-x}$, (ii) $(2-3x^2)^{\frac{1}{2}}$, (iii) $\log_e(2x-x^2)$,

(iv) $\tan 4x$, (v) xe^{-3x}, (vi) $\sin^{-1}\left(\dfrac{x}{2a}\right)$.

2. A lighted candle is raised above a horizontal table. Find the height for maximum illumination of the surface at a given point, whose distance from the point vertically under the candle is a feet.

[If A is the flame, P the point, θ the angle between AP and the normal to the table at P, the intensity of illumination varies directly as $\cos\theta$ and inversely as AP^2.]

3. The roof of a house is inclined at an angle 60° to the horizontal and the height of the wall is h feet. A ladder whose length is l feet rests with one end on the ground at a distance y feet from the foot of the wall and the other end on the roof at a distance x feet from the top of the wall. If the inclination of the ladder to the ground is θ, prove $\dfrac{dy}{dx} = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\tan\theta$.

4. Evaluate the following integrals:

(i) $\int_4^9 x^{\frac{1}{2}} dx$, (ii) $\int_{-1}^0 \frac{1}{2-3x} dx$, (iii) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos(1-x)\, dx$,

(iv) $\int_0^1 x(1-x^2)^{\frac{1}{2}} dx$, (v) $\int_1^2 e^{1-3x} dx$, (vi) $\int_0^1 \frac{x}{1+x^2} dx$.

5. A thin uniform rod OA, 6 feet long, swings in a vertical plane about a horizontal axis through one end O. Express the velocity of a point in the rod distant r feet from O at the instant when the rod makes an angle θ with the vertical, in terms of $\frac{d\theta}{dt}$. Hence write down an approximate expression for the kinetic energy of a small portion, length Δr, of the rod, and find by integration the total kinetic energy of the rod at this instant. If the rod falls from the horizontal position, shew that when OA is vertical, the velocity of A is 24 feet/sec. nearly.

H.

1. Find $\frac{dy}{dx}$ for the following values of y:

(1) $(x^3+1)(x^2-2x+7)$... (2 ways),

(2) $\frac{x^3+1}{x^2-2x+7}$, (3) $(2x-3)(3x^2-x+1)(4x^3-2x^2-3)$,

(4) $\tan 5x$, (5) $\frac{x}{\sqrt{1-x^2}}$,

(6) $\sec^2 3x$, (7) $x(a^2+x^2)\sqrt{a^2-x^2}$.

2. Find the maximum and minimum values of

$$\sin x - \sin 2x \text{ between } x=0 \text{ and } x=2\pi.$$

Sketch the curve $y = \sin x - \sin 2x$.

3. OA, AB are the crank and connecting rod of a steam engine, OA rotating about O and B sliding along a fixed line through O. If

$$OA=a, \quad AB=c, \quad OB=r, \quad \angle BOA=\theta,$$

express $\cos\theta$ in terms of r, a, c and shew that $\frac{dr}{d\theta} = -\frac{ar\sin\theta}{r-a\cos\theta}$.

OA $=3\frac{1}{2}''$, AB $=12''$, $\angle$ OAB $=90°$, angular velocity of OA $=100$ revolutions per minute; find velocity of B.

4. The motion of the needle of a galvanometer is given by the equation $\theta = 6e^{-\frac{1}{2}t}\sin 3t$, where θ is the angle in radians made by the needle with the zero position at the end of t seconds. Find the angular velocity of the needle at time t and shew that the extreme excursions to the right and left of the zero position occur at intervals of $\frac{\pi}{3}$ seconds and that the angles corresponding to these extreme excursions form a G. P. of common ratio $-e^{-\frac{\pi}{6}}$.

5. The equation of a curve is $y = b\sin^2\frac{\pi x}{a}$. Find the mean height of that portion for which x lies between 0 and a.

I.

1. If
$$y = ae^{3x}\sin\left(\frac{2x}{3} + \frac{\pi}{4}\right),$$
find
$$\frac{dy}{dx} \text{ and } \frac{d^2y}{dx^2},$$
and shew that
$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + \frac{85}{9}y = 0.$$

2. A crank CP revolves uniformly n times a minute and the other end A of a rod PA moves to and fro in the straight line AC.

If CP $= a$ ft., PA $= 2a$ ft., CA $= x$ ft., shew that

(1) $\sin\theta = 2\sin\phi$,

(2) $x = a\cos\theta + 2a\cos\phi$,

where θ is $\angle$ ACP and ϕ is $\angle$ CAP.

Hence get $\frac{d\phi}{dt}$ and $\frac{dx}{dt}$ in terms of θ and ϕ.

What are the angular velocity of AP and the speed of A when $n = 40$, $a = 1$, $\theta = 39°$?

3. On the same sheet trace the curves
$$y = \log_e(1 + x), \quad y = x - \frac{x^2}{2}$$
between $x = 0$ and $x = 1$, using as large a scale as possible. Find the slope of each at the points where $x = 0$, ·1, ·2, ·4, ·6, ·8, 1.

4. Find the work done in slowly compressing a gas from a volume of 10 c. ins. to 6 c. ins., the initial pressure being atmospheric and the temperature being kept constant. Take atmospheric pressure to be 14·75 lbs. per sq. inch. [$pv=$const.]

If the compression takes place under adiabatic conditions, i.e. so rapidly that no heat is lost, find the work done. [In this case $pv^{1.41}=$constant.]

5. Draw a figure to shew that

$$\int_0^a f(x)\,.\,dx=\int_0^a f(a-x)\,dx.$$

Hence prove

$$\int_0^{\frac{\pi}{2}} \sin^2 x\,dx=\int_0^{\frac{\pi}{2}} \cos^2 x\,.\,dx=\frac{1}{2}\int_0^{\frac{\pi}{2}} 1\,.\,dx=\frac{\pi}{4}.$$

J.

1. (i) If
$$x^2-y^2=a^2,$$
prove
$$\frac{dy}{dx}=\frac{x}{y}.$$

Hence prove that if the normal at any point P on $x^2-y^2=a^2$ meet the x-axis in G, PO$=$PG.

(ii) Shew that
$$y=-\frac{\sin nx}{n^2-1}$$
is a solution of
$$\frac{d^2y}{dx^2}+y=\sin nx.$$

2. Given $\log_e 6=1{\cdot}7918$, find $\log_e 6{\cdot}1$.

3. A chain hangs in the form given by the equation

$$y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right) \text{ or } y=c\cosh\frac{x}{c},$$

the axis of y being vertical.

Find the area of the figure enclosed between the curve and the chord which joins the lowest point of the curve to the point where $x=2c$.

4. Find the C.G. of a circular segment whose arc subtends an angle 2α at the centre of a circle of radius r.

Find also the C.G. of a sector of angle 2α and of an arc of angle 2α.

5. The sides of a triangular lamina are 3, 4, 5 feet. Find the radius of gyration about the axis of revolution of the solid formed by the revolution of the triangle about the longest side.

K.

1. If $p=ab^{\frac{\theta}{m+n\theta}}$ where a, b, m, n are constants [De la Roche's formula connecting vapour pressure and absolute temperature], shew that

$$\frac{dp}{d\theta}=\frac{mp\log b}{(m+n\theta)^2}.$$

2. If the equation of a curve is $xy+y+x=0$, find the equations of the tangent and normal at the point where $x=4$.

3. P moves in a straight line AB and O is a fixed point. Shew that the angular velocity of P about O at any instant is $\frac{v\sin\theta}{r}$, where r ft. is the length OP, v ft./sec. the velocity of P at the instant, and θ the angle between PO and AB.

4. If a plane area [A sq. ft.] revolve about an axis in its plane which does not intersect it, the volume generated is Al c. ft., where l ft. is the length of the path traced by the C.G. of the area. [Theorem of Pappus, end of 3rd century A.D.]

5. Find the area included between the curve

$$x^2y=x^3+a^3,$$

the x-axis and the ordinates $x=a$, $x=2a$.

Find also the co-ordinates of the C.G. of this area.

L.

1. If $$y^2+3\mathrm{A}x-\mathrm{A}^2=0,$$
where A is a constant, prove

$$4\left(\frac{dy}{dx}\right)^2+18\frac{x}{y}\left(\frac{dy}{dx}\right)=9.$$

2. If $(x-a)^n$ is a factor of $f(x)$, shew that $(x-a)^{n-1}$ is a factor of $f'(x)$.

Hence given that $$3x^3-7x^2-8x+20=0$$
has two equal roots, solve the equation.

3. OA, OB are the bounding radii of a quadrant of a circle, PQRS is a rectangle having one corner (P) in OA, one (Q) in OB and two (R, S) in the arc AB. Find the maximum area of this rectangle. [Work in terms of $\angle$COR, C being the mid-point of arc AB.]

4. A point in a certain mechanism moves so that its position at the end of t seconds is given by

$$x+4=3\cos\frac{\pi t}{3}+\cos\frac{2\pi t}{3},$$

$$y=\sin\frac{\pi t}{3}.$$

Find the position, velocity and acceleration of the point when $t=2$. [Distances are in feet.]

5. The x-axis is taken along the axis of the cylinder of an engine sc that the piston moves between $x=0$ and $x=10$ [unit 1 inch]. The thrust on the piston is 81 lbs. wt. from $x=0$ to $x=6$ and $1000x^{-1 \cdot 4}$ lbs. wt. from $x=6$ to $x=10$. Find the work done in one stroke in ft. lbs. wt.

Also draw the graph shewing the relation between the thrust and x, and get the work done by counting squares in the appropriate area.

M.

1. Find the differential coefficients with respect to x of

$$\tan^2 3x,\quad x^2\log 4x,\quad \sqrt{\frac{a-2x}{a+3x}},\quad (5x+3)^{10}(2x^2+7)^{12},$$

$$(a^2+x^2)^{\frac{3}{2}},\quad \tan^{-1}\left(\frac{3x}{a}\right).$$

2. A point moves in a plane so that at end of t secs. its coordinates are $x=t$, $y=\sin 2\pi t$, the unit being 1″. Draw the path from $t=0$ to $t=2$. Find the angle the direction of motion of the point makes with the x-axis at time $t=0 \cdot 2$. Find area contained between x-axis and the path from $t=0$ to $t=0 \cdot 5$.

3. A and B are two sources of heat, 20 feet apart. The intensity of A is twice that of B. Find the position between them where the combined heating effect is least.

4. If
$$V=RC+L\frac{dC}{dt},$$
and
$$C=100\sin 600t,$$
R being 2 and L being $0 \cdot 005$, find V in the form
$$V=A\sin(600t+q).$$

5. A point is moving over a straight line of length $2a$ with simple harmonic motion, period T seconds. Find its speed and acceleration at a time t seconds from rest. If V be the maximum value of v, shew that

$$\int_0^T v^2 dt=\tfrac{1}{2}V^2T.$$

N.

1. Find equations of tangent and normal to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (h, k).

Find the distances from the origin at which this tangent and normal cut the x-axis and the product of these distances.

2. A ladder has one end on the ground and the other projects over a vertical wall 10 feet high. The lower end slides along the ground with uniform speed 1 foot per second. Find the angular velocity of the ladder when its foot is 10 feet from the wall, the motion being supposed to take place in a plane perpendicular to the wall.

3. A straight road runs along the edge of a common and a person on the common, at a distance of 1 mile from the nearest point (A) of the road, wishes to go to a distant place (C) on the road in the least time possible. If his rates of walking on common and road are 4 and 5 miles an hour respectively, prove that he must strike the road at a point B, $1\frac{1}{3}$ miles from A.

4. Evaluate (i) $\int_0^1 \cos(3-2x)\,dx$, (ii) $\int_0^{\frac{\pi}{2}} \sin x \,.\, \cos^2 x\,dx$,

(iii) $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x\,dx$, (iv) $\int_0^{\frac{\pi}{30}} \sin x\,.\,dx$, (v) $\int_0^{\cdot 1047} x\,dx$.

Account for the fact that the last two results are approximately equal.

5. CA is a radius of a circle, O its mid-point. OB perpendicular to CA meets the circumference in B. The head of a shell is formed by the revolution of the arc AB about BO. Find its volume if $\text{CA}=2a$.

[If P be any point on the arc, work in terms of $\angle\text{ACP}\ (=\theta)$.]

O.

1. Find $\frac{dy}{dx}$ if

(i) $2x^3+3x^2y+y^3+5xy+7x+8y+15=0$,

(ii) $x^2+\sin y+y^2=0$,

(iii) $y=3u^2-7u+2$ where $u=2x^3+3x+2$,

(iv) $\sin x=\log_e y$.

Find (v) $\frac{d\,(3y^2-7y+11)}{d\,(3y)}$, (vi) $\frac{dx}{d\sin x}$, (vii) $\frac{d\sin x}{d\cos x}$.

2. (i) If $y=x\log x$, shew that y is least when $x=\dfrac{1}{e}$.

(ii) If $y=\dfrac{\log x}{x}$, shew that y is greatest when $x=e$.

3. (i) Shew that $v\dfrac{dv}{ds}$ is acceleration.

(ii) If $v^2=as+b$, prove that the acceleration is constant.

(iii) If $v^2=as^2+bs+c$, prove that the acceleration varies as the distance from a certain fixed point in the line of motion.

(iv) If $s=\dfrac{1}{2}vt$, prove that the acceleration is constant.

4. A curve whose equation is $xy=$constant passes through the point (4, 5). Find the area of the figure bounded by the curve, the axis of x and the two ordinates $x=4$, $x=12$.

Find also the position of the centre of gravity of this area and the radius of gyration of the area about the x-axis.

5. Find the mean value between $\theta=0$ and $\theta=\dfrac{\pi}{2}$ of

(i) $\sin\theta$, (ii) $\cos\theta$, (iii) $\sin\theta\cos\theta$,
(iv) $\sin^2\theta$, (v) $\cos^2\theta$.

P.

1. Find $\dfrac{dy}{dx}$ when y is

(i) $x^2\sin x$, (ii) x^2e^{2x}, (iii) $e^{ax}\cos^3 bx$,
(iv) $\dfrac{\tan x}{x}$, (v) $\sqrt{x\sin x}$. (vi) $\sqrt{\dfrac{1+\cos x}{2}}$.
(vii) $\sqrt{\sin\sqrt{x}}$.

2. A line is drawn through (1, 3) cutting the axes in A and B respectively, so that OA . OB is a minimum. Find the equation of the line.

3. A body mass m lbs. is moving in a straight line. If at the end of t secs. its displacement from some standard position is s ft., its speed v ft./sec. and its acceleration a ft./sec.², and if k foot-poundals be the kinetic energy, shew that

$$\frac{dk}{ds}=ma \text{ and } \frac{dk}{dt}=ma\,.\,v.$$

4. If a piston-rod drives a crank of length r through a connecting-rod of length l, prove that its acceleration at the end of the stroke is $\omega^2 r\left(1 \pm \frac{r}{l}\right)$, where ω is the uniform angular velocity of the crank.

5. A cart weighing 200 lbs. containing 400 lbs. of sand, ascends a straight hill rising 40 feet vertically altogether. The sand is assumed to be thrown out uniformly so that the cart reaches the top empty. Find by integration the whole work done against gravity in the ascent.

Q.

1. Give equations of tangent to $x^2+2y^2=5$ at each of the points where $y=1$.

2. AB is a straight line; P, Q two given points on opposite sides of it, PM, QN are perpendicular to AB. PM $=p$ ft., QN $=q$ ft., MN $=a$ ft. R is any point in AB such that MR $=x$ ft. If the speed of light on the P-side of AB is u ft./sec. and on the Q-side v ft./sec., find the time light would take to travel from P to Q by the path PRQ. If R′ be the position of R for which this time is the least and if PR′, QR′ make angles θ, ϕ with the normal to AB, prove

$$\frac{\sin\theta}{\sin\phi}=\frac{u}{v}.$$

3. P is any point (h, k) on the curve $y=be^{\frac{x}{a}}$. PN is the ordinate of P, PT and PG the tangent and normal meeting OX in T and G.

Prove NT constant and $\qquad \text{NG}=\dfrac{k^2}{a}.$

4. In a tangent galvanometer the current is proportional to the tangent of the angle of deflection ($c=k\tan\theta$). Find the approximate error in the inferred value of the current if an error $\Delta\theta$ is made in reading the deflection. Shew that the relative error in the current $\left(\frac{\Delta c}{c}\right)$ is approximately $\frac{2\Delta\theta}{\sin 2\theta}$, and for a given error in the deflection is least when $\theta=45°$.

5. Prove that the area common to the two parabolas

$$y^2=4a(a+x), \qquad y^2=4b(b-x),$$

is

$$\frac{8}{3}(a+b)\sqrt{ab},$$

and find the C. G. of this area.

R.

1. The equation $$s = ae^{-bt}\sin\frac{2\pi t}{T}$$
gives the displacement (s feet) of the end of a stiff spring from the position of equilibrium at the end of t seconds.

If $a=10$, $T=\cdot 9$, $b=\cdot 75$, find the speed and acceleration of the end of the spring at the end of 5 seconds.

2. Shew that $\frac{V^2}{54} - 3\frac{V-12}{V+12}$ has a minimum value when $V=6$ and that it has no maximum value.

3. In a simple horizontal engine the crank is r feet and the connecting-rod l feet long. At any instant the crank and connecting-rod make angles θ and ϕ respectively with the line of dead centres. If the angular velocity of the crank be constant and equal to ω, shew that the angular velocity of the connecting-rod is
$$\omega\,.\frac{r\cos\theta}{l\cos\phi} = \frac{\omega}{l}\,.\,RP,$$
where P is the end of the crank and R the point where the connecting-rod meets the perpendicular to the line of dead centres through the centre of the crank circle.

4. Find (i) $\int_0^{\frac{\pi}{2}} \sin 3x\cos 2x\,dx$, (ii) $\int_0^{\frac{\pi}{4}} \cos 3x\sin 2x\,dx$,

(iii) $\int_0^{\pi} \cos 3x\cos 2x\,dx$, (iv) $\int_0^{2\pi} \sin 3x\sin 2x\,dx$;

and shew that if m and n are whole numbers
$$\int_0^{\pi} \sin mx\cos nx\,dx = 0$$
or $\frac{2m}{m^2-n^2}$ according as $(m+n)$ and $(m-n)$ are both even or both odd.

Shew also that
$$\int_0^{\pi} \sin mx\sin nx\,dx = \int_0^{\pi} \cos mx\cos nx\,dx = 0.$$

5. Draw roughly $y=\sin x$ from $x=0$ to $x=\frac{\pi}{2}$.

Consider the area bounded by the curve, the x-axis and the ordinate $x=\frac{\pi}{2}$.

Divide the base into 10 equal parts. Shew that the area lies between ·919 and 1·08.

Divide the base into n equal parts each $\frac{\pi}{2n}$ or h.

Shew that the area lies between

$$\frac{h}{\sin\frac{h}{2}}\sin\frac{\pi}{4}\sin\left(\frac{\pi}{4}-\frac{h}{2}\right) \quad \text{and} \quad \frac{h}{\sin\frac{h}{2}}\sin\frac{\pi}{4}\sin\left(\frac{\pi}{4}+\frac{h}{2}\right).$$

Then shew that when h is indefinitely diminished the limit of each of these is

$$2\sin^2\frac{\pi}{4} \text{ or } 1.$$

Compare with what you get by finding $\int_0^{\frac{\pi}{2}} \sin x\,dx$ in the ordinary way.

[Assume $\quad \sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\dots$ to n terms

$$=\frac{\sin\frac{n\beta}{2}\sin\left(\alpha+\frac{n-1}{2}\beta\right)}{\sin\frac{\beta}{2}}.]$$

S.

1. (i) If $$q=ae^{2t}+be^{3t},$$

shew that $$\frac{d^2q}{dt^2}-5\frac{dq}{dt}+6q=0.$$

(ii) If $$q=e^{3t}(a\cos 4t+b\sin 4t),$$

shew that $$\frac{d^2q}{dt^2}-6\frac{dq}{dt}+25q=0.$$

2. OX, OY are two lines at right angles to each other. A, B are two points 5 feet apart in a rod which is constrained to move so that A is always in OX and B in OY. Supposing A to move along OX at a uniform rate of 1 inch per second, find the rate of motion of B along OY and the angular velocity of AB at the instant when OB is 3 ft.

3. The co-ordinates of any point on a cycloid being given in the form

$$x=a(\theta-\sin\theta),$$
$$y=a(1-\cos\theta),$$

shew that the whole area bounded by one complete arch and the line on which the generating circle rolls is $3\pi a^2$.

4. A submarine cable consists of a circular core surrounded by a concentric circular covering. The speed of signalling through it varies as $\dfrac{\log x}{x^2}$ where x is the ratio of the radius of the covering to that of the core.

Find x so that the speed may be a maximum.

5. Find the centre of pressure of a circular area (radius r) with its plane vertical and its centre at a distance c below the surface ($c > r$).

T.

1. Find (i) $\displaystyle\int \frac{x^3+a^3}{2x} \cdot dx$, (ii) $\displaystyle\int \cot x\,dx$,

(iii) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 2x\,dx$, (iv) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$.

(v) $\displaystyle\int x^2(3x^3+1)^{\frac{1}{2}}\,dx$.

2. Water is poured at a constant rate into a conical glass, which is filled in 2 minutes, the height of the cone being 12 inches. At what rate in inches per minute is the surface of water rising (1) when the glass is filled to half its height, (2) when half the liquid has been poured in?

3. The equation to the Folium of Descartes is

$$x^3+y^3=3axy.$$

Shew that any point given by

$$x=\frac{3am}{1+m^3}, \qquad y=\frac{3am^2}{1+m^3}$$

lies on the curve.

Find $\dfrac{dy}{dm}$ and $\dfrac{dx}{dm}$, and hence $\dfrac{dy}{dx}$.

Find the equations to the tangents at the points where $m=\frac{1}{2}$ and $m=1$. Draw the curve carefully for positive values of m taking $a=1$. [Unit 1″.]

Find $\dfrac{dy}{dx}$ in the above otherwise.

4. If $s = \mathrm{A}\sin nt + \mathrm{B}\cos nt$, where A, B, n are constants, find the maximum kinetic energy, the space-average and the time-average of the kinetic energy for one journey between the extreme positions and shew that the three results are proportional to 6, 4, 3.

5. The cross-section of an I girder has the following dimensions:

Top Flange	6×2 ins.
Bottom Flange	15×2 ins.
Web	18×2 ins.

Find the radius of gyration of this section about an axis in its plane through its C.G. perpendicular to the web.

CHAPTER XII

APPROXIMATE SOLUTION OF EQUATIONS

268. SUPPOSE we wish to solve

$$\mathbf{x^3 - 6x^2 + 9x - 1 = 0.}$$

The ordinary graphical method is to draw the graph of

$$y = x^3 - 6x^2 + 9x - 1,$$

and find where it cuts the x-axis.

The degree of accuracy attained will depend on the care expended in drawing the graph and on the size of the scales used. We have seen that a knowledge of the gradients at different points is a great help in drawing the graph.

Put $\qquad f(x) = x^3 - 6x^2 + 9x - 1,$

then $\qquad f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3).$

Make a table of values:

x	0	1	2	3	4
$f(x)$	-1	3	1	-1	3
$f'(x)$	9	0	-3	0	9

If we draw the graph (Fig. 133) the roots appear to lie between (i) 0 and ·2, (ii) 2·3 and 2·4, (iii) 3·5 and 3·6.

269. If we want better approximations, it is better to work at each root separately.

Take the first root, that is the one lying between 0 and ·2.

x	0	·2
$f(x)$	-1	·568
$f'(x)$	9	6·72

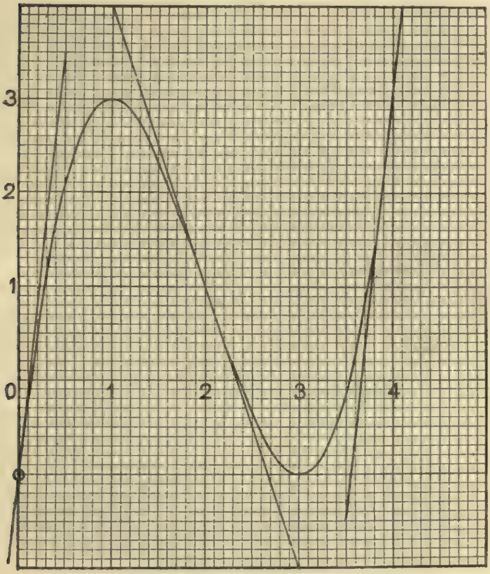


Fig. 133.

P, Q (Fig. 134) are the points $(0, -1)$ and $(·2, ·568)$. PS, QT are the lines through these points with gradients 9 and 6·72, i.e. they are the tangents at P and Q.

It is evident from the figure without actually drawing the curve that it cuts the axis very nearly where $x = ·12$.

We find $$f(·12) = -·004672.$$

If we want something nearer, the figure suggests trying ·13.

$$f(·13) = ·070797.$$

∴ the root is between ·12 and ·13.

x	·12	·13
$f(x)$	$-$·00467	·07080
$f'(x)$	7·60	7·49

P′ is (·12, $-$·00467), Q′ is (·13, ·07080).

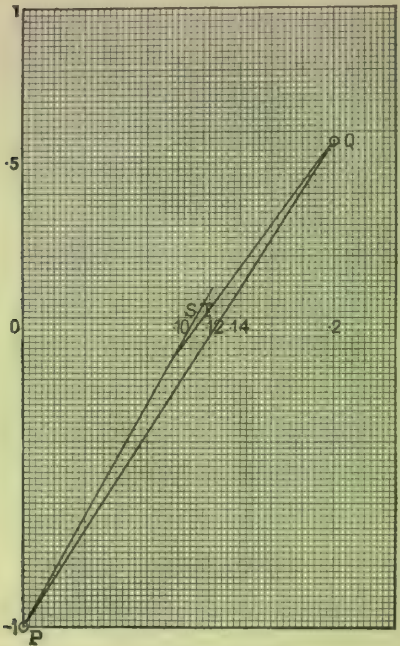


Fig. 134.

It will be found, even on a large scale drawing (Fig. 135), difficult to distinguish between P′Q′ and the tangents at P′ and Q′.

·1205 seems the best approximation.

As a matter of fact $f(·1205) = -·0009$ nearly.

270. In the example just worked out $f(0)=-1$, $f(1)=3$, i.e. the point on the graph of $y=f(x)$ corresponding to $x=0$ is below, while that corresponding to $x=1$ is above the x-axis, therefore we might say that the graph of $y=f(x)$ cuts the x-axis at some intermediate point.

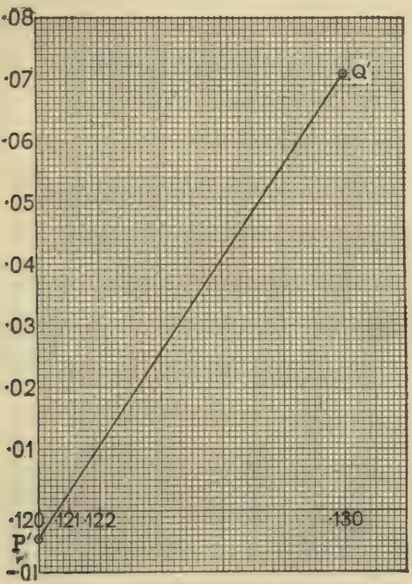


Fig. 135.

It may happen that $y=f(x)$ cuts the x-axis at more than one point between the points corresponding to $x=0$ and $x=1$, e.g. referring to Fig. 133,

$$f(0)=-1,\ f(4)=+3,$$

and there are 3 roots between 0 and 4.

$$f(1)=3,\ f(3)=-1,$$

and there is 1 root between 1 and 3.

Also if $f(x)$ has the same sign for two different values of x, say a and b, it does not follow that there is no root between a and b; for instance, taking the same figure, $f(2) = 1$, $f(4) = 3$ and there are 2 roots between 2 and 4.

If $f(a)$ and $f(b)$ have opposite signs, there is an odd number of roots between a and b. If $f(a)$ and $f(b)$ have the same sign, there is no root or an even number of roots between a and b.

271. There is an exceptional case if $f(x)$ becomes infinite between a and b.

e.g. suppose $$f(x) = x^2 + \frac{1}{x}.$$

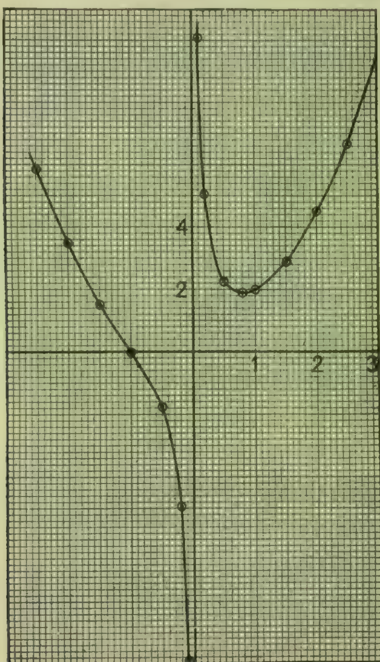


Fig. 136.

Here $$f(-\cdot 5) = -1{\cdot}75,$$
$$f(\cdot 5) = 2{\cdot}25,$$
but there is no root between $-\cdot 5$ and $\cdot 5$.

A glance at Fig. 136 will shew the reason of this.

When $x = 0$, y is infinite.

When x is very small and negative, y is very large and negative.

e.g. if $$x = -\frac{1}{10^6}, \quad y = -10^6 + \frac{1}{10^{12}}.$$

When x is very small and positive, y is very large and positive.

e.g. if $$x = \frac{1}{10^6}, \quad y = 10^6 + \frac{1}{10^{12}},$$

so that as x passes through the value 0, y suddenly changes from a very large negative value to a very large positive value.

It thus appears that if $f(a)$ and $f(b)$ have opposite signs we must take care that $f(x)$ does not become infinite for some value of x between a and b, otherwise our inference that there is a root between a and b may be wrong.

272. If $f(a)$ and $f(b)$ have opposite signs and if moreover $f'(x)$ and $f''(x)$ do not change sign between $x = a$ and $x = b$, then between $x = a$ and $x = b$ the graph of $y = f(x)$ will have one of the shapes shewn in § 94, and will cut the x-axis only once between $x = a$ and $x = b$.

273. A method of continual approximation to the value of a root will now be considered which does not require the actual drawing of the graph.

Taking the same equation as before, we have
$$f(x) = x^3 - 6x^2 + 9x - 1,$$
$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3),$$
$$f''(x) = 6(x-2).$$
Now $f(2) = 1$, $f(3) = -1$.

Also from $x=2$ to $x=3$, $f'(x)$ is negative and $f''(x)$ is positive.

$\therefore$ the curve $y=f(x)$ between $x=2$ and $x=3$ is like Fig. 137, and there is only one root between 2 and 3.

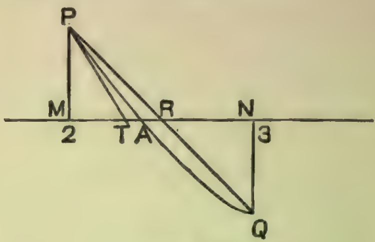


Fig. 137.

If PT is the tangent at P and the chord PQ cuts the x-axis in R, the point A where the curve cuts the x-axis is between T and R.

Now $f'(2) = -3$.

$\therefore$ equation of PT is $y-1=-3(x-2)$.

Putting $y=0$, $x=2+\frac{1}{3}$. [This is OT where O is the origin.]

This value of x [$2\frac{1}{3}$] is a better approximation to the root than 2.

Again gradient of PQ is $-\dfrac{2}{1}$, and equation of PQ is

$$y-1=-2(x-2).$$

Putting $y=0$, $x=2+\frac{1}{2}$. [This is OR.]

This value of x [$2\frac{1}{2}$] is a better approximation to the root than 3.

$\therefore$ we know that the root is between $2\frac{1}{3}$ and $2\frac{1}{2}$ and therefore between 2·33 and 2·50.

[We must be careful here not to overstate the smaller or to understate the larger value, for instance we cannot safely say that the root is between 2·34 and 2·50.]

To get a closer approximation, use the points P′, Q′ corresponding to $x = 2{\cdot}33$ and $x = 2{\cdot}50$ exactly as we have just used P and Q.

$$f(2{\cdot}33) = 0{\cdot}045937, \quad f(2{\cdot}5) = -0{\cdot}375,$$
$$f'(2{\cdot}33) = -2{\cdot}6733.$$

Equation P′T′ $\quad y - {\cdot}045937 = -2{\cdot}6733\,(x - 2{\cdot}33).$

$\therefore$ at T′ $\qquad x = 2{\cdot}33 + \dfrac{{\cdot}045937}{2{\cdot}6733} = 2{\cdot}33 + ({\cdot}017 +).$

Equation P′Q′ $\quad y - {\cdot}045937 = -\dfrac{{\cdot}420937}{{\cdot}17}(x - 2{\cdot}33)$

$$= -2{\cdot}4761\,(x - 2{\cdot}33).$$

$\therefore$ at R′ $\qquad x = 2{\cdot}33 + \dfrac{{\cdot}045937}{2{\cdot}4761} = 2{\cdot}33 + ({\cdot}019 -).$

$\therefore$ the root lies between $2{\cdot}347$ and $2{\cdot}349$.

If we want a still closer approximation we use the points P″, Q″ corresponding to $x = 2{\cdot}347$ and $x = 2{\cdot}349$.

$$f(2{\cdot}347) = 0{\cdot}000781923, \quad f(2{\cdot}349) = -0{\cdot}004491451,$$
$$f'(2{\cdot}347) = -2{\cdot}638773.$$

Equation P″T″

$$y - {\cdot}000781923 = -2{\cdot}638773\,(x - 2{\cdot}347).$$

$\therefore$ at T″

$$x = 2{\cdot}347 + \frac{{\cdot}000781923}{2{\cdot}638773} = 2{\cdot}347 + ({\cdot}0002963 +).$$

Equation P″Q″

$$y - {\cdot}000781923 = -2{\cdot}636687\,(x - 2{\cdot}347).$$

$\therefore$ at R″

$$x = 2{\cdot}347 + \frac{{\cdot}000781923}{2{\cdot}636687} = 2{\cdot}347 + ({\cdot}0002966 -).$$

$\therefore$ the root lies between **2·3472963** and **2·3472966**.

274. We might have "closed in" on the root more rapidly by drawing first the portion of the graph of

$$y = x^3 - 6x^2 + 9x - 1,$$

which lies between $x=2$ and $x=3$ from the data

$$f(2)=1, \qquad f(3)=-1,$$
$$f'(2)=-3, \qquad f'(3)=0.$$

This would shew quite clearly that the root lies between 2·3 and 2·4.

In working out the smallest root, we might proceed graphically as far as the stage reached in § 269. Fig. 135 seems to shew that the root is between ·1205 and ·1207. Also $f'(x)$ is positive and $f''(x)$ negative so that the curve is like Fig. 40 (b), § 94. Proceeding on the lines of § 273, we find that the root lies between ·12061474 and ·12061477.

275. Sometimes it may not be so easy as in the example just considered to decide whether $f'(x)$ keeps the same sign between the values of x chosen, and it may be useful to argue as follows:

$$f'(x)=3x^2-12x+9,$$
$$f''(x)=6x-12.$$

Between $x=2$ and $x=3$, $f''(x)$ is clearly positive.

i.e. $f'(x)$ continually increases; but $f'(2)=-3$ and $f'(3)=0$.

$\therefore$ $f'(x)$ is negative all the way.

276. The rough figure (137) should be drawn, so that there shall be no doubt at which end of the arc the tangent ought to be drawn.

The following example will shew what may happen if we neglect this precaution.

Suppose
$$\left.\begin{aligned} f(x)&=3x^3-8x+1\\ f'(x)&=9x^2-8\\ f''(x)&=18x \end{aligned}\right\}.$$

Here
$$f(1)=-4, \qquad f(2)=9,$$

and between $x=1$ and $x=2$, $f'(x)$ and $f''(x)$ do not change sign; they are both positive.

$\therefore$ there is 1 root between 1 and 2.

Now if we start from P (the point on $y=f(x)$ corresponding to $x=1$), we have since $f'(1)=1$:

Equation of tangent PT is $y+4=1\,(x-1)$.

$\therefore$ at T, $x=5$.

But 5 is not a better approximation to the root than 1.

Since $f'(x)$ and $f''(x)$ are both positive between $x=1$ and $x=2$ the curve is of the form shewn in Fig. 138 and a glance at the figure will shew that we should have worked from Q (corresponding to $x=2$).

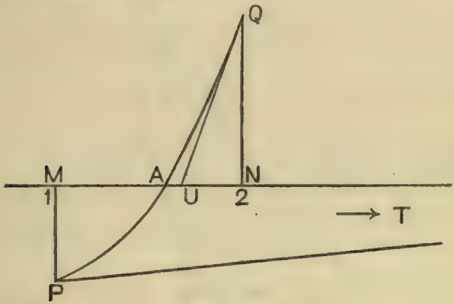


Fig. 138.

U is nearer to A than N is.

T is not nearer to A than M is.

Of course with a curve of this form it may happen that T comes nearer to A than M does, but on the other hand it may not, whereas U must be nearer to A than N is.

What we ought to do in this case is to consider 2 as the first approximation to the root and get a better one by finding the co-ordinates of U.

277. As another example take the equation $\cos x = x$ and let it be required to find the least positive root.

An inspection of the tables shews that x is between [42°] and [43°] where [42°] stands for the C.M. of 42°.

If
$$\left.\begin{aligned} f(x) &\text{ is } x - \cos x \\ f'(x) &= 1 + \sin x \\ f''(x) &= \cos x \end{aligned}\right\}.$$

Working with 5-figure tables

$$\left.\begin{aligned} f[42^\circ] &= \cdot 73304 - \cdot 74314 = - \cdot 01010 \,(\pm \cdot 00001) \\ f[43^\circ] &= \cdot 75049 - \cdot 73135 = + \cdot 01914 \,(\pm \cdot 00001) \end{aligned}\right\}.$$

Also between $x = [42^\circ]$ and $x = [43^\circ]$ $f'(x)$ and $f''(x)$ are both positive and the curve is like Fig. 139.

$$f'[43^\circ] = 1\cdot 68200 \,(\pm \cdot 000005).$$

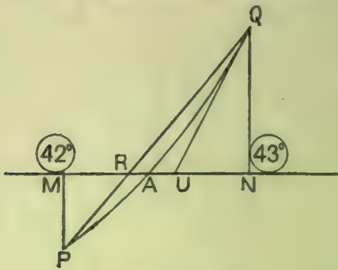


Fig. 139.

Equation QU is

$$y - \cdot 01914 = 1\cdot 68200\,(x - [43^\circ]).$$

$$\therefore \text{ at U} \quad x = [43^\circ] - \frac{\cdot 01914\,*}{1\cdot 68200} = [43^\circ] - (\cdot 01137 +).$$

Equation PQ is

$$\begin{aligned} y - \cdot 01914 &= \frac{\cdot 02924}{[1^\circ]}\,(x - [43^\circ]) \\ &= \frac{\cdot 02924}{\cdot 01745}\,(x - [43^\circ]) \\ &= 1\cdot 6756\,(x - [43^\circ]). \end{aligned}$$

$$\therefore \text{ at R} \quad x = [43^\circ] - \frac{\cdot 01914\,*}{1\cdot 6756} = [43^\circ] - (\cdot 01143 -).$$

* We must be careful to understate the value of the first of these fractions, and it will be safe to take it as $\frac{\cdot 01913}{1\cdot 682005}$. Similarly the value of the other fraction should be overstated.

$\therefore$ x lies between $\cdot 73906$ and $\cdot 73912$, or since $\cdot 01137$ and $\cdot 01143$ each differ from [39′] by less than [1′], we may say x is approximately the C.M. of 42° 21′.

As a matter of fact for this value of x

$$f(x) = \cdot 73914 - \cdot 73904 = \cdot 0001.$$

EXERCISES. LXXXIV.

1. Shew that $x^3 - x - 1 = 0$ has one and only one root between 1 and 2, and then shew by the method of § 273 that this root lies between 1·1 and 1·6.

2. $4x^5 - 24x^3 + 2 = 0$ has a root near 3. Shew that 2·7 is a better approximation.

3. $x^4 - 2x - 1 = 0$ has a root between -1 and 0. Shew that $-\cdot 3$ is a better approximation than 0 and $-\cdot 7$ than -1.

4. $3x^4 - 40x^3 + 1200x - 2900 = 0$ has a root between 3 and 4. Shew that it is between 3·3 and 3·6.

5. $x^3 - 3x^2 + 2x - 5 = 0$ has a root near 2·9. Shew that it is between 2·9041 and 2·9042.

6. Shew that $x^3 - 2x - 5 = 0$ has a root between 2 and 3, and find its value correct to 3 decimal places.

7. Find all three roots of $20x^3 - 24x^2 + 3 = 0$ correct to 3 decimal places.

8. Shew that $x^3 - 5x + 2 = 0$ has one and only one root between 0 and 1, and find its value correct to 4 decimal places.

9. Shew that $3x^3 = x + 1$ has only one real root and find its value correct to 5 decimal places.

10. Shew that $x^3 - 30x^2 + 2600 = 0$ has a root between 12 and 13 and find its value correct to 2 decimal places. [First write for x, $y + 10$.]

11. Shew that if a floating sphere of radius r'' and specific gravity s, sink in water to a depth x'', then

$$x^3 - 3rx^2 + 4r^3s = 0.$$

If the sphere be of wood (sp. gr. ·6) and its radius be 10″, to what depth will it sink? [Correct to nearest hundredth of an inch.]

12. $x^3-10x^2+40x-35=0$ has a root between 1 and 2. Find its value correct to 4 decimal places.

13. Find the real roots of $x^4-3x-5=0$, each correct to 3 significant figures.

14. A plane is drawn parallel to the base of a hemisphere of 1 foot radius, dividing it into two parts of equal volume. Find its distance from the centre correct to the nearest thousandth of an inch.

15. $x-\sin x=\cdot 02$ is satisfied if x is the C.M. of some angle near 30°. Find the angle so that your error does not exceed 1 minute.

16. Solve approximately $\tan x=2x$ between $x=0$ and $x=\frac{\pi}{2}$. Your answer must not be more than ·001 in error.

17. Solve $\tan x=-\frac{1}{2}x$ between 0 and π correct to 3 decimal places.

18. Shew from a graph that the line $3y=2x+1$ cuts the curve $y=\tan x$ roughly where $x=\frac{\pi}{4}$.

Find the value of x at the point of intersection correct to 3 decimal places.

19. Solve the equation $\log_e x=\frac{1}{3}x$ correct to 3 decimal places.

20. Sketch the shape of the curve $y=x^2\sin x$: suppose x to range from 0 to π.

Determine approximately the maximum ordinate within this range.

CHAPTER XIII

SOME METHODS OF INTEGRATION

Further examples in integration.

278. Our success in evaluating integrals of the form $\int f(x)\,dx$ has so far depended on our ability to recognise $f(x)$ as the result of differentiating some standard function.

e.g. we know that

$$\frac{d\,(\tan x)}{dx} = \sec^2 x,$$

and hence that

$$\int \sec^2 x\,dx = \tan x + c.$$

It will be convenient at this stage to collect the most important standard results. They have all been worked out either in the text or in the exercises. It is understood that an arbitrary constant should be added to every function in the column headed $\int z\,dx$.

When a more complicated expression is to be integrated, our object will be to reduce it to one or other of these standard forms. A few of the simpler methods of effecting such reduction are given in this chapter, and it will be seen that many functions yield to these methods of treatment, but no rule can be given for the integration of any given function.

y	$\dfrac{dy}{dx}$	z	$\displaystyle\int z\,dx$
x^n	nx^{n-1}	x^n except when $n=-1$	$\dfrac{1}{n+1}x^{n+1}$
$\sin x$	$\cos x$	$\cos x$	$\sin x$
$\cos x$	$-\sin x$	$\sin x$	$-\cos x$
$\tan x$	$\sec^2 x$	$\sec^2 x$	$\tan x$
$\cot x$	$-\operatorname{cosec}^2 x$	$\operatorname{cosec}^2 x$	$-\cot x$
$\sec x$	$\sec x \tan x$	$\sec x \tan x$	$\sec x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x$
$\sin^{-1}\dfrac{x}{a}$	$\dfrac{1}{\sqrt{a^2-x^2}}$	$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\dfrac{x}{a}$
$\tan^{-1}\dfrac{x}{a}$	$\dfrac{a}{a^2+x^2}$	$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$
e^x	e^x	e^x	e^x
$\log_e x$	$\dfrac{1}{x}$	$\dfrac{1}{x}$	$\log_e x$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$
$\cosh x$	$\sinh x$	$\sinh x$	$\cosh x$
$\tanh x$	$\operatorname{sech}^2 x$	$\operatorname{sech}^2 x$	$\tanh x$
$\left.\begin{array}{c}\sinh^{-1}\dfrac{x}{a}\\ \text{or } \log_e\dfrac{x+\sqrt{x^2+a^2}}{a}\end{array}\right\}$	$\dfrac{1}{\sqrt{x^2+a^2}}$	$\dfrac{1}{\sqrt{x^2+a^2}}$	$\left\{\begin{array}{c}\sinh^{-1}\dfrac{x}{a}\\ \text{or } \log\dfrac{x+\sqrt{x^2+a^2}}{a}\end{array}\right.$
$\left.\begin{array}{c}\cosh^{-1}\dfrac{x}{a}\\ \text{or } \log_e\dfrac{x+\sqrt{x^2-a^2}}{a}\end{array}\right\}$	$\dfrac{1}{\sqrt{x^2-a^2}}\left(\dfrac{x}{a}>1\right)$	$\dfrac{1}{\sqrt{x^2-a^2}}$ $\left(\dfrac{x}{a}>1\right)$	$\left\{\begin{array}{c}\cosh^{-1}\dfrac{x}{a}\\ \text{or } \log\dfrac{x+\sqrt{x^2-a^2}}{a}\end{array}\right.$
$\left.\begin{array}{c}\tanh^{-1}\dfrac{x}{a}\\ \text{or } \frac{1}{2}\log_e\dfrac{a+x}{a-x}\end{array}\right\}$	$\dfrac{a}{a^2-x^2}\ (x^2<a^2)$	$\dfrac{1}{a^2-x^2}$ $(x^2<a^2)$	$\left\{\begin{array}{c}\dfrac{1}{a}\tanh^{-1}\dfrac{x}{a}\\ \text{or } \dfrac{1}{2a}\log_e\dfrac{a+x}{a-x}\end{array}\right.$

y	$\frac{dy}{dx}$	z	$\int z\,dx$
$\coth^{-1}\frac{x}{a}$ or $\tanh^{-1}\frac{a}{x}$ or $\frac{1}{2}\log_e\frac{x+a}{x-a}$	$-\frac{a}{x^2-a^2}\ (x^2>a^2)$	$\frac{1}{x^2-a^2}$ $(x^2>a^2)$	$-\frac{1}{a}\coth^{-1}\frac{x}{a}$ or $-\frac{1}{a}\tanh^{-1}\frac{a}{x}$ or $-\frac{1}{2a}\log_e\frac{x+a}{x-a}$
$\log_e\sin x$	$\cot x$	$\cot x$	$\log_e\sin x$
$\log_e\cos x$	$-\tan x$	$\tan x$	$-\log_e\cos x$ or $\log_e\sec x$
$\log_e(\sec x+\tan x)$ or $\log_e\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$	$\sec x$	$\sec x$	$\log_e(\sec x+\tan x)$ or $\log_e\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$
$\log_e(\operatorname{cosec} x+\cot x)$ or $-\log\tan\frac{x}{2}$	$-\operatorname{cosec} x$	$\operatorname{cosec} x$	$-\log_e(\operatorname{cosec} x+\cot x)$ or $\log_e\tan\frac{x}{2}$

279. Corresponding to the formula of differentiation

$$\frac{df(u)}{dx}=\frac{df(u)}{du}\cdot\frac{du}{dx} \text{ or } f'(u)\frac{du}{dx},$$

we have the formula of integration

$$\boldsymbol{\int\phi(\mathrm{u})\frac{d\mathrm{u}}{d\mathrm{x}}\,.\,d\mathrm{x}=\int\phi(\mathrm{u})\,.\,d\mathrm{u},}$$

for since $\quad f'(u)\,.\,\frac{du}{dx}=\frac{df(u)}{dx}$ and $\frac{df(u)}{du}=f'(u)$,

$$\int f'(u)\,.\,\frac{du}{dx}\,.\,dx=f(u) \text{ and } f(u)=\int f'(u)\,du,$$

or $\quad \displaystyle\int\phi(u)\,.\,\frac{du}{dx}\,.\,dx=\int\phi(u)\,.\,du,$

where $\phi(u)$ stands for $f'(u)$.

That is to say $\frac{du}{dx}\,.\,dx$ occurring under the integral sign may be replaced by du.

Ex. 1. $$\int \sin^3 x \cos x\,dx = \int \sin^3 x \frac{d(\sin x)}{dx} \,.\, dx$$
$$= \int \sin^3 x\, d(\sin x) = \int u^3 du \text{ [where } u \text{ is } \sin x]$$
$$= \frac{u^4}{4} = \frac{\sin^4 x}{4}.$$

Everything here depends on noticing that the expression to be integrated ($\sin^3 x \cos x$) is the product of a function of $\sin x$ (viz. $\sin^3 x$) and $\cos x$, which is the differential coefficient of $\sin x$.

Generally if the expression, $f(x)$, to be integrated can be written in the form $\phi(u) \,.\, \frac{du}{dx}$ we can say that

$$\int f(x) \,.\, dx = \int \phi(u) \,.\, du,$$

and it may happen that $\int \phi(u) \,.\, du$ is easier to get than $\int f(x) \,.\, dx$.

Ex. 2. Find $$\int \frac{2x+3}{\sqrt{x^2+3x+5}} dx.$$

Here we notice that $(2x+3)$ is $\frac{d(x^2+3x+5)}{dx}$ and thus $(2x+3)\,dx$ may be replaced by $d(x^2+3x+5)$ and the integral may be written

$$\int \frac{d(x^2+3x+5)}{\sqrt{x^2+3x+5}}$$

or $$\int \frac{du}{\sqrt{u}} \text{ where } u \text{ is } (x^2+3x+5),$$

which is $2\sqrt{u}$ or $2\sqrt{x^2+3x+5}$.

Ex. 3. $$\int \sin^5 x \,.\, dx = \int \sin^4 x \,.\, \sin x\, dx = \int -(1-\cos^2 x)^2\, d(\cos x)$$
$$= -\int (1-u^2)^2\, du = -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) \text{ where } u = \cos x$$
$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x.$$

Ex. 4. $$\int \frac{dx}{\sqrt{3x^2 - x + 1}}.$$

The expression under the $\sqrt{}$

$$= 3\left(x - \frac{1}{6}\right)^2 + \frac{11}{12} = \left\{\sqrt{3}\left(x - \frac{1}{6}\right)\right\}^2 + \frac{11}{12},$$

and $$\frac{d\sqrt{3}\left(x - \frac{1}{6}\right)}{dx} = \sqrt{3}.$$

$\therefore$ integral is $$\frac{1}{\sqrt{3}}\int \frac{du}{\sqrt{u^2 + \frac{11}{12}}},$$

where u stands for $\sqrt{3}\left(x - \frac{1}{6}\right)$,

$$= \frac{1}{\sqrt{3}} \sinh^{-1} \frac{u}{\sqrt{\frac{11}{12}}} = \frac{1}{\sqrt{3}} \sinh^{-1} \frac{6x - 1}{\sqrt{11}},$$

or $$\frac{1}{\sqrt{3}}\log_e\left(u + \sqrt{u^2 + \frac{11}{12}}\right) = \frac{1}{\sqrt{3}}\log_e\left\{\sqrt{3}\left(x - \frac{1}{6}\right) + \sqrt{3x^2 - x + 1}\right\}.$$

EXERCISES. LXXXV.*

Find $\int z\,dx$ for the following values of z:

1. $\sin x \cos^5 x$. **2.** $x \sin x^2$. **3.** $\dfrac{2x+3}{x^2+3x+5}$.

4. $\dfrac{1}{x^2+3x+5}$. **5.** $\dfrac{2x+4}{x^2+3x+5}$. **6.** $\dfrac{4x+11}{x^2+3x+5}$.

7. $\dfrac{1}{x}\sin(\log_e x)$. **8.** $\dfrac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$. **9.** $\dfrac{1}{x(1+\log_e x)}$.

10. $\dfrac{1}{9-4x^2}$ $(4x^2<9)$. **11.** $\dfrac{1}{4x^2-9}$ $(4x^2>9)$. **12.** $\dfrac{1}{9+4x^2}$.

13. $\cos^3 x$. **14.** $\tan^5 x \sec^2 x$. **15.** $\tan^3 x$.

* In this and following Exs. results should be checked by differentiation.

16. $\sqrt{2a+x}$. **17.** $\sin^3 x \cos^2 x$.

18. $\dfrac{1}{x^2-3x+2}$. [(i) x^2-3x+2 pos. (ii) x^2-3x+2 neg.]

19. $\dfrac{3x+1}{x^2+1}$. **20.** $\tan(ax+b)$. **21.** $\sin x \cos 2x$.

22. $\dfrac{x}{\sqrt{2-3x^2}}$. **23.** $\dfrac{2-x}{3-x^2}[x^2<3]$. **24.** $\dfrac{x^2}{x^3+1}$.

25. $\dfrac{\sin x}{\cos^5 x}$. **26.** $x\sqrt{a^2-x^2}$. **27.** $\dfrac{\log_e(x^2)}{x}$.

28. $\dfrac{1}{(1-x)\sqrt{x}}(x<1)$. **29.** $\dfrac{\tan^{-1}x}{1+x^2}$. **30.** $\dfrac{\log_e(2x)}{x}$.

Integration by Substitution.

280. This depends on the same principle as the last method, viz. that

$$\int \phi(u) \cdot \frac{du}{dx} \cdot dx = \int \phi(u) \cdot du.$$

Ex. 1. Find $$\int \sqrt{a^2-x^2} \cdot dx.$$

Put $x = a\sin\theta$; then $\sqrt{a^2-x^2} = a\cos\theta$, and

$$\frac{dx}{d\theta} = a\cos\theta,$$

so that $\int \sqrt{a^2-x^2}\, dx$ becomes $\int a\cos\theta \cdot d(a\sin\theta)$,

$$= \int a\cos\theta \cdot a\cos\theta\, d\theta$$

$$= \frac{a^2}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)$$

$$= \frac{1}{2}\left[a^2 \sin^{-1}\frac{x}{a} + x\sqrt{a^2-x^2}\right].$$

Ex. 2. Find $\int x\sqrt{2x+5}\,.\,dx.$

Put $2x+5=z, \;\therefore\; x=\dfrac{z-5}{2}$, and $\dfrac{dx}{dz}=\dfrac{1}{2}$,

so that
$$\int x\sqrt{2x+5}\,.\,dx=\int\frac{z-5}{2}\sqrt{z}\,.\,\frac{1}{2}\,dz$$
$$=\frac{1}{4}\int\left(z^{\frac{3}{2}}-5z^{\frac{1}{2}}\right)dz$$
$$=\frac{1}{4}\left[\frac{2}{5}z^{\frac{5}{2}}-\frac{10}{3}z^{\frac{3}{2}}\right]$$
$$=\frac{z^{\frac{3}{2}}}{60}\left[6z-50\right]$$
$$=\frac{1}{15}(2x+5)^{\frac{3}{2}}(3x-5).$$

EXERCISES. LXXXVI.

Find:

1. $\displaystyle\int\frac{dx}{(a^2+x^2)^2}\,[x=a\tan\theta]$.

2. $\displaystyle\int\frac{x^2+2x+3}{(x-3)^5}\,dx\,[x-3=z]$.

3. $\displaystyle\int\frac{dx}{(x^2-a^2)^2}\,[x^2>a^2]$.

4. $\displaystyle\int\sqrt{9-4x^2}\,.\,dx$.

5. $\displaystyle\int x^2\sqrt{a^2-x^2}\,.\,dx$.

6. $\displaystyle\int x^3\sqrt{a^2-x^2}\,.\,dx$.

7. $\displaystyle\int\frac{x^2+5}{(x+5)^2}\,.\,dx$.

8. $\displaystyle\int\sqrt{a^2+x^2}\,.\,dx\,[x=a\sinh\theta]$.

9. $\displaystyle\int\sqrt{x^2-a^2}\,dx\,[x=a\cosh\theta]$.

10. $\displaystyle\int\frac{1}{x\sqrt{a^2-x^2}}\,dx$.

11. $\displaystyle\int\frac{x^3}{x-2}\,dx\,[x-2=z]$.

12. $\displaystyle\int\frac{x^3}{2x-3}\,dx$.

13. $\displaystyle\int_0^a\frac{x^3}{2a-x}\,dx$.

14. $\displaystyle\int\frac{dx}{\sqrt{x^2+a^2}}\,(x=a\sinh\theta)$.

15. By means of the substitution $\tan x=\sinh\phi$, or $\sec x=\cosh\phi$, shew that $\int\sec^3 x\,dx=\int\cosh^2\phi\,.\,d\phi$, and hence evaluate $\int\sec^3 x\,dx$. (See Exs. LXXXVII. 15.)

Integration by Parts.

281. This is an inversion of the rule for differentiating a product.

Ex. $$\frac{d(x\sin x)}{dx} = \sin x + x\cos x \ldots\ldots\ldots\ldots\ldots(1).$$

If one of the members of the right-hand side can be integrated so can the other.

(1) may be written

$$\frac{d(x\sin x)}{dx} = -\frac{d\cos x}{dx} + x\cos x.$$

$$\therefore\ x\cos x = \frac{d(x\sin x + \cos x)}{dx},$$

or $$\int x\cos x\,dx = x\sin x + \cos x.$$

Similarly by differentiating $x\cos x$ get $\int x\sin x\,.\,dx$,

and " $x\log x$ get $\int \log x\,.\,dx$.

282. The work can be arranged more conveniently.

We have $$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx},$$

$$\therefore\ uv = \int u\,dv + \int v\,du,$$

or $$\int \mathbf{u\,dv} = \mathbf{uv} - \int \mathbf{v\,du},$$

and it may happen that $\int v\,du$ is easier to get than $\int u\,dv$.

Ex. 1. $$\int x\cos x\,dx = \int x\,d(\sin x) = x\sin x - \int \sin x\,dx$$
$$= x\sin x + \cos x.$$

We might have said

$$\int x \cos x\, dx = \int \cos x \,.\, d\left(\frac{x^2}{2}\right)$$

$$= \frac{x^2}{2} \cos x - \int \frac{x^2}{2}\, d(\cos x)$$

$$= \frac{x^2}{2} \cos x + \int \frac{x^2}{2} \sin x \,.\, dx,$$

but this leads to an integral $\int \frac{x^2}{2} \sin x\, dx$ which is no easier to get than the original one $\int x \cos x\, dx$.

Ex. 2. $$\int \log_e x\, dx = x \log_e x - \int x\, d(\log_e x)$$

$$= x \log_e x - \int x \,.\, \frac{1}{x}\, dx$$

$$= x \log_e x - x.$$

EXERCISES. LXXXVII.

Find:

1. $\int x \log_e x\, dx.$
2. $\int xe^x dx.$
3. $\int x^2 e^x\, dx.$
4. $\int \tan^{-1} x dx.$
5. $\int \sin^{-1} x\, dx$
6. $\int x \sin x\, dx.$
7. $\int x \sin mx \cos nx\, dx.$
8. $\int x^3 \log_e x\, dx.$
9. $\int \frac{\log_e x}{x^3}\, dx.$
10. $\int x \sec^2 x\, dx.$

11. Shew that

$$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx,$$

and
$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx.$$

Hence get
$$\int e^x \sin x\, dx = \frac{1}{2} e^x (\sin x - \cos x),$$

and
$$\int e^x \cos x\, dx = \frac{1}{2} e^x (\sin x + \cos x).$$

12. Similarly get

$$\int e^{ax} \sin (bx + c)\, dx \quad \text{and} \quad \int e^{ax} \cos (bx + c)\, dx.$$

13. Shew that

$$\int \sin^5 x\, dx = -\sin^4 x \cos x + \int 4 \sin^3 x \cos^2 x\, dx.$$

Hence
$$5 \int \sin^5 x\, dx = -\sin^4 x \cos x + 4 \int \sin^3 x\, dx.$$

Similarly shew

$$n \int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \,.\, dx,$$

and
$$n \int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \,.\, dx.$$

14. Shew that

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx.$$

Hence find
$$\int_0^{\frac{\pi}{2}} \sin^7 x\, dx \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin^8 x\, dx.$$

15. Shew that

$$\int \sec^3 x\, dx = \sec x \tan x - \int \tan^2 x \sec x\, dx.$$

Hence
$$2 \int \sec^3 x\, dx = \sec x \tan x + \int \sec x \,.\, dx.$$

MISCELLANEOUS EXERCISES ON CHAPTER XIII.

Find $\int z\,dx$ for the following values of z:

1. $\dfrac{5x+1}{x^2+4}$. **2.** $x\tan^{-1}x$. **3.** $\dfrac{x^3+x}{x+1}$.

4. $\dfrac{x^2}{1+x^6}$. **5.** $\dfrac{\log_e ax}{x}$. **6.** $\dfrac{\cos x}{1+\sin x}$.

7. $\dfrac{1}{x^2\sqrt{a^2-x^2}}$. **8.** $\dfrac{x^3}{\sqrt{a^2+x^2}}$. **9.** $\dfrac{1}{\sqrt{2x^2+3x+5}}$.

10. $\dfrac{x^2-5}{2\sqrt{x}}$. **11.** $\dfrac{x^2-5}{2\sqrt{x-5}}$. **12.** $\sin^3 2x\cos 2x$.

13. $x\sin 3x$. **14.** $x^2\sin 3x$. **15.** $(3-2x)^{\frac{2}{3}}$.

16. $x(3-2x)^{\frac{2}{3}}$. **17.** $\dfrac{x}{(2-3x)^2}$. **18.** $\tan\left(\dfrac{\pi x}{4}+a\right)$.

19. $\dfrac{2-x}{3-x^2}$. **20.** $\dfrac{4x^3-3x}{(x-2)^3}$. **21.** $\operatorname{cosec} x \left[\tan\dfrac{x}{2}=t\right]$.

22. $\sec x$. **23.** $\dfrac{1}{\sqrt{a^2-x^2}}\ (x=a\sin\theta)$.

24. $\dfrac{1}{a^2+x^2}\ (x=a\tan\theta)$.

Evaluate the following definite integrals:

25. $\displaystyle\int_0^{\frac{\pi}{4}}\sec^5 x\,dx$. **26.** $\displaystyle\int_1^2\frac{dx}{5-x^2}$. **27.** $\displaystyle\int_3^4\frac{dx}{5-x^2}$.

28. $\displaystyle\int_0^{\frac{\pi}{2}}\cos^7 x\,dx$. **29.** $\displaystyle\int_0^{\frac{\pi}{4}}\sin^5 x\,dx$. **30.** $\displaystyle\int_0^{\frac{\pi}{2}}\cos^8 x\,dx$.

31. $\displaystyle\int_0^{\frac{\pi}{2}}\frac{dx}{5+4\cos x}$. $\left[\text{Put } \tan\dfrac{x}{2}=t.\right]$ **32.** $\displaystyle\int_0^{\frac{\pi}{2}}\frac{dx}{4+5\cos x}$.

33. O is the lowest point of the chain of a suspension bridge, P any other point on the chain whose co-ordinates are (x, y) referred to horizontal and vertical axes through O. If the resultant load on the portion OP of the chain is wx lbs. (w being constant) and if T, T_0 lbs. wt. are the pulls a P and O respectively, shew that

$$\frac{dy}{dx} = \frac{w}{T_0}x,$$

and hence get the equation of the curve of the chain.

Also shew that $$T = T_0\sqrt{1 + \frac{w^2}{T_0^2}x^2}.$$

34. A body mass m lbs. starts with a speed of u ft./sec. and moves horizontally, the resistance to motion being Kv^2 poundals, where K is a constant, and v ft./sec. its speed at any instant. Shew that the speed at the end of t seconds is $\dfrac{mu}{m + Kut}$ ft./sec.

35. Assume that the resistance to the movement of a ship through the water is of the form $(a^2 + b^2v^2)$ lbs. wt., where v is the speed and a, b are constants, m lbs. being the mass of the ship. If the engines are stopped, find a formula for the time in which the speed falls to one-half of its original amount u.

36. A body of mass m lbs. falls from rest vertically in a medium in which the resistance is Kv^2 lbs. wt. where K is constant and v ft./sec. is the speed.

Shew that

$$t = \frac{v_1}{2g}\log\frac{v_1 + v}{v_1 - v}, \text{ or } \frac{v_1}{g}\tanh^{-1}\frac{v}{v_1}, \text{ where } v_1 = \sqrt{\frac{m}{K}}.$$

Hence get v in terms of t and v_1, and shew that

$$s = \frac{v_1^2}{g}\log\left(\cosh\frac{gt}{v_1}\right).$$

37. A block slides in a straight line OA. The resistance to its movement when it is at a distance x ft. from O is $R \cdot \dfrac{a^2}{a^2 + x^2}$ lbs. wt. where OA is a ft.

Find the work done in pushing the block from A to O.

38. A train is moving on the level, the power exerted by the engine being constant and the total resistance proportional to the speed. Shew that $\frac{dv}{dt} = \frac{a}{v} - bv$, where a and b are constants, and hence that $a - bv^2 = \mathsf{A}e^{-2bt}$, where A is a constant.

If the constant power is 400 H.P. and if the resistance is 12 lbs. wt. per ton at 40 miles an hour, and if the mass of the train be 200 tons, find the values of the constants a, b, A; t being reckoned from the instant at which the speed is 40 miles an hour. Shew that the speed can never reach 50 miles an hour. Find the speed after half-a-minute and after half-an-hour. [$g = 32$.]

CHAPTER XIV

POLAR CO-ORDINATES

283. We have hitherto considered the position of a point in a plane as determined by its distances from two fixed axes. These distances are the "Cartesian co-ordinates" of the point.

There is another system in which the position of a point is determined by a distance and an angle. The position is obviously determined if we know the length, direction and sense of the line joining the point to some fixed point. For instance if A is a given point and we say that B is 100 yards N. 30° E. from A, B is determined. This is the principle of "Polar Co-ordinates."

284. O is a fixed point (fig. 140), OX a fixed line through

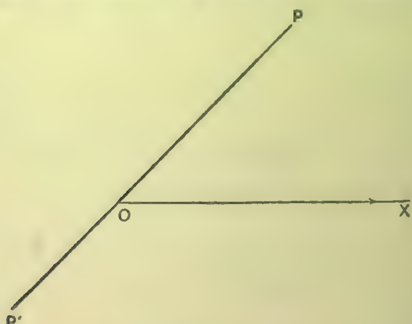


Fig. 140.

O. The position of P is known if we are given the *length* OP and the *angle* XOP, through which a line must swing to get from the position OX to the position OP. These are called the "polar co-ordinates" of P, and are denoted by (r, θ). O is called the pole and OX the "*initial line.*"

285. There is an infinite number of ways of writing the co-ordinates of a point P, e.g. if $XOP = 30^\circ$ and OP is 3 inches, taking 1″ as the unit of length we may give the co-ordinates of P as

$$(3, 30^\circ), (3, 390^\circ), (3, -330^\circ) \text{ etc.}$$

286. If r is negative the radius bounding the angle θ must be produced backwards through O, e.g. if the point is $(-3, 210^\circ)$, first draw OP′ so that $XOP' = 210^\circ$, then produce P′O backwards through O so that OP is 3″. This brings us to the same point P as in § 285.

EXERCISES. LXXXVIII.

1. Taking 1″ as the unit of length, plot the points $(3, 40^\circ)$, $(-3, 140^\circ)$, $(-3, 220^\circ)$, $(3, 320^\circ)$, $(-3, -50^\circ)$, $(3, -500^\circ)$, $(-3, -1000^\circ)$.

2. Draw the loci:

(i) $r=a$, (ii) $\theta=40^\circ$, (iii) $r=a\cos\theta$, (iv) $r\cos\theta=a$, (v) $r=a\sin\theta$, (vi) $r\sin\theta=a$.

287. Figure 141 shews part of the curve $r=f(\theta)$, P is the point (r, θ) and Q the point $(r+\Delta r, \theta+\Delta\theta)$, PS is perpendicular to OQ, and R is the point where a circle, centre O radius OP, cuts OQ.

$\therefore$ $POQ = \Delta\theta$ and $RQ = \Delta r$.

Now $\cot PQR = \dfrac{QS}{SP} = \dfrac{QR+RS}{SP} = \dfrac{QR}{SP} + \dfrac{RS}{SP}$ and by sufficiently diminishing $\Delta\theta$ we can make $\dfrac{RS}{SP}$ as small as we like (§ 32); also

$\frac{QR}{SP} = \frac{\Delta r}{r \sin \Delta\theta} = \frac{\Delta r}{r . \Delta\theta} \cdot \frac{\Delta\theta}{\sin \Delta\theta}$ and by sufficiently diminishing $\Delta\theta$ we can bring this as near to $\frac{1}{r} \cdot \frac{dr}{d\theta}$ as we please.

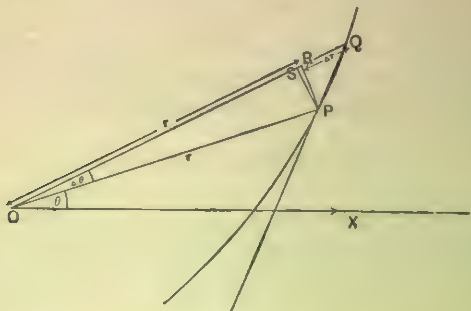


Fig. 141.

$\therefore$ if ϕ be the angle between OP and the tangent at P (fig. 142) $\cot \phi = \frac{1}{r} \cdot \frac{dr}{d\theta}$ or $\tan \phi = r \cdot \frac{d\theta}{dr}$.

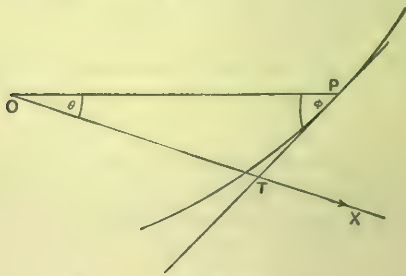


Fig. 142.

Ex. If

$$r = a(1 - \cos\theta),$$

$$\frac{dr}{d\theta} = a \sin\theta.$$

$$\therefore \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \cot\frac{\theta}{2}.$$

$$\therefore \phi = \frac{\theta}{2}.$$

e.g. if $\theta = 90°$, $\phi = 45°$; if $\theta = 120°$, $\phi = 60°$; if $\theta = 60°$, $\phi = 30°$. [See fig. 143.]

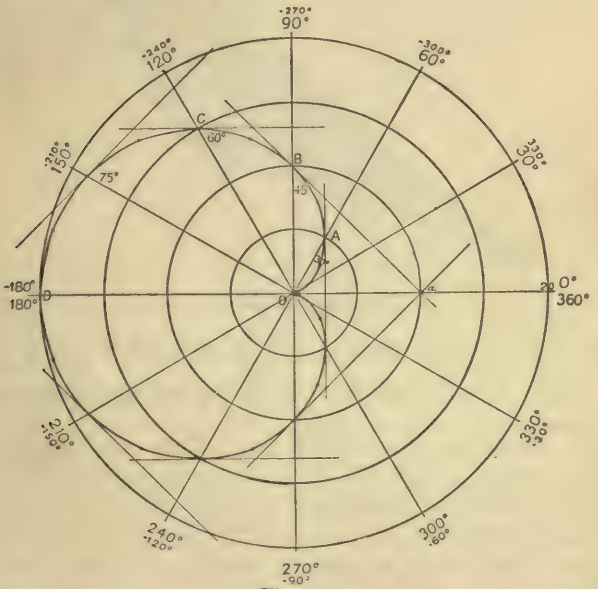


Fig. 143.

These facts may be found useful in drawing the curve, just as the knowledge of the gradient was a help in drawing a curve whose equation was given in Cartesian co-ordinates.

288. Areas.

Suppose we want the area bounded by HK, part of the curve $r = f(\theta)$, and the radii OH, OK.

Divide $\angle$ HOK into any number of equal parts and with O as centre describe circular arcs as shewn in fig. 144.

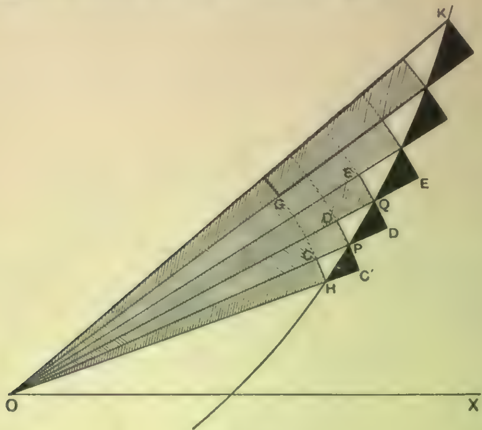


Fig. 144.

The area required lies between the sum of the internal sectors OHC, OPD, OQE, ... and the sum of the external sectors OPC′, OQD′, ...

The difference between these sums is the sum of the areas CHC′P, DPD′Q ..., and this is the same as the area GK. By sufficiently diminishing the angle of the sectors or increasing the number of the sectors, we can make this area GK as small as we like, and thus the area required is the limit of the sum of either the internal or external sectors as the number of sectors is indefinitely increased or the angle of the sectors indefinitely diminished [cp. § 136].

If P be any point (r, θ) on the arc HK and if Q be $(r + \Delta r, \theta + \Delta\theta)$, the area of the sector POD is $\frac{1}{2} r^2 . \Delta\theta$, and area

$$\text{HOK} = \underset{\Delta\theta \to 0}{\text{Lt}} \sum_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} r^2 \Delta\theta = \int_\alpha^\beta \frac{1}{2} r^2 d\theta,$$

where α and β are the angles XOH, XOK respectively.

289. *Ex.* If the curve be $r = a\,(1 - \cos\theta)$ [Fig. 143],

$$\text{area OAB} = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2} r^2 d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{a^2}{2}\left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2}\left[\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{4} - 2 + \frac{7\sqrt{3}}{8}\right]$$

$$= \cdot 1505 a^2.$$

EXERCISES. LXXXIX.*

1. Find values of r corresponding to $\theta = 0, 20°, 40°, 60°, 80°, 90°, \ldots 360°$; if $r = a\,(1 - \cos\theta)$.

Also find the values of ϕ corresponding to $\theta = 0, 60°, 90°, 120°, 150°, 180°$. Draw the graph of $r = a\,(1 - \cos\theta)$, using these values. (See Fig. 143.)

2. Find $\tan\phi$ if

(i) $r = a\sin\theta$, (ii) $r = a\cos\theta$, (iii) $r\cos\theta = a$, (iv) $r\sin\theta = a$.

Interpret the results geometrically.

3. Draw graphs of

(i) $r = a\sin 2\theta$, (ii) $r = a\cos 2\theta$, (iii) $r = a\sin 3\theta$,

(iv) $r = a\cos 3\theta$, (v) $r\sin 2\theta = a$, (vi) $r\sin 3\theta = a$,

(vii) $\frac{a}{r} = 1 + \cos\theta$, (viii) $\frac{a}{r} = 1 + \frac{1}{2}\cos\theta$, (ix) $\frac{a}{r} = 1 + 2\cos\theta$,

(x) $r^2 = a^2\cos 2\theta$, (xi) $r = 3 + 2\cos\theta$.

4. Find the area enclosed by $r = a\cos\theta$.

* Paper specially ruled for polar co-ordinates can be obtained. Lines are drawn through O (see Fig. 143) at intervals of 5°, and concentric circles are described at small equal distances.

5. Find the area of one loop of $r=a\sin 2\theta$.

6. Find the area of a loop of $r^2=a^2\cos 2\theta$.

7. Draw the curve $r=a\theta$. (The spiral of Archimedes.)

8. In the curve $r=ae^{k\theta}$, shew that ϕ is constant.

[This is called the equiangular or logarithmic spiral.]

9. Draw the curves (i) $r=ae^{\theta}$, (ii) $r=ae^{2\theta}$.

10. Shew that $r=6e^{\theta}$ may be written $r=e^{\theta+\log_e 6}$ and hence that $r=6e^{\theta}$ is $r=e^{\theta}$ turned round the pole through a certain angle.

Generally shew that $r=ae^{k\theta}$, $r=be^{k\theta}$ represent the same spiral in different positions.

ANSWERS

EXERCISES I. p. 3.

1. (i) 39·39 m./hr. (ii) 57·77 ft./sec. **2.** 42 m./hr.

3. (i) $2\frac{8}{11}$ m./hr. (ii) 80 yds./min.

4. ·0146 ft./sec. **5.** ·43 m./hr.

EXERCISES II. p. 6.

1. 44 ft./sec.; $46\frac{2}{3}$ ft./sec.; 50·77 ft./sec. A's is best. 22 ft.; $23\frac{1}{3}$ ft.; 25·4 ft. A's.

2. [All in chs./min.] 31, 39, 57, 52, 59, 58, 62, $62\frac{1}{2}$. (i) 59, (ii) 62.

3. 50, 57·36, 42·26, 54·46, 45·40, 51·50, 48·48 cms. 8·832 ± ·006, 9·288 ± ·006, 8·92 ± ·01, 9·20 ± ·01, 9·00 ± ·03, 9·12 ± ·03 cms./sec. Between 8·97 and 9·12 cms./sec.

4. 173·21, 142·81, 214·45, 153·99, 196·26, 166·43, 180·40 cms. 36·480 ± ·012, 49·488 ± ·012, 38·44 ± ·02, 46·10 ± ·02, 40·68 ± ·06, 43·14 ± ·06 cms./sec. Between 41·4 and 42·3 cms./sec.

EXERCISES III. p. 9.

1. $(99 - 24h + 2h^2)$ ft./sec., $\left(99 + \frac{h^2}{2}\right)$ ft./sec.

2. 42, 68, 54·25, 46·72, 44·33, 42·2303, 42·023003 ft. 26, 24·5, 23·6, 23·3, 23·03, 23·003 ft./sec. $(42 + 23h + 3h^2)$ ft.; $(23 + 3h)$ ft./sec.; 23 ft./sec.; $(23 - 3h)$ ft./sec.; 23 ft./sec.; $\{23 + 3(n - m)\}$ ft./sec.

3. (i) 26 ft./sec. (ii) 23, 29, 26 ft./sec. (iii) 26 ft./sec.

4. (i) 333 ft./sec. (ii) $330\frac{3}{4}$ ft./sec. (iii) $337\frac{1}{2}$ ft./sec.

5. 249 ft., $(249 + 253h + 96h^2 + 16h^3 + h^4)$ ft., 253 ft./sec.

EXERCISES IV. p. 18.

1. (i) $605\frac{1}{4}$. (ii) 604. (iii) 603. (iv) 602. **2.** ·87; ·86; about ·87.

3. 6; $2a$. **4.** 3; 75; $3a^2$. **5.** (i) 6π. (ii) 2π. (iii) 3.

6.

r	4	5	4·5	4·2	4·1	4·01	$4+h$
S	64π	100π	81π	$70{\cdot}56\pi$	$67{\cdot}24\pi$	$64{\cdot}3204\pi$	$(64+32h+4h^2)\pi$
V	$\frac{256}{3}\pi$	$\frac{500}{3}\pi$	$\frac{364{\cdot}5}{3}\pi$	$\frac{296{\cdot}352}{3}\pi$	$\frac{275{\cdot}684}{3}\pi$	$\frac{257{\cdot}924804}{3}\pi$	$\frac{256+192h+48h^2+4h^3}{3}\pi$

(i) $\frac{244}{3}\pi$, $\frac{217}{3}\pi$, $\frac{201{\cdot}76}{3}\pi$, $\frac{196{\cdot}84}{3}\pi$, $\frac{192{\cdot}4804}{3}\pi$, $\frac{192+48h+4h^2}{3}\pi$.

(ii) 36π, 34π, $32{\cdot}8\pi$, $32{\cdot}4\pi$, $32{\cdot}04\pi$, $(32+4h)\pi$.

(iii) $\frac{244}{3}\pi$ etc. c.c./sec. [see (i)]. (iv) 36π etc. sq. cms./sec. [see (ii)].

(v) 64π. (vi) 32π. (vii) 64π c.c./sec. (viii) 32π sq. cms./sec.

7. (i) 20. (ii) $16+4h$. (iii) 16.

EXERCISES V. p. 29.

1. 9, 12·25, 9·61, 9·0601, $9+6h+h^2$, 8·41, $9-6h+h^2$;
6·5, 6·1, 6·01, $6+h$, 5·9, $6-h$; 6; $\tan^{-1}6=80°\ 32'$.

2. 6. **3.** $\tan^{-1}1{\cdot}2=50°\ 12'$. **4.** $12+6h+h^2$; 12.

5. $12+h^2$; $12+6(n-m)+(n^2+nm+m^2)$. **6.** 0, 3, 27, 48.

7. $\tan^{-1}4{\cdot}8=78°\ 14'$. **8.** 8, 0. **9.** 4; 4; 4.

10. 3; $3+8h+6h^2-h^4$; 8; 0.

EXERCISES VI. p. 37.

1. ·109. **2.** 27 ft./sec.

EXERCISES VIII. p. 51.

1. (i) $v=32t$. (ii) $v=100-32t$. (iii) $v=3t^2$. (iv) $v=5$.
(i) 96, 160, 544. (ii) 4, −60, −444. (iii) 27, 75 867.
(iv) 5, 5, 5 (ft./sec).

2. 64 ft./sec.; 6·4 ft.; 2·44 %; ·64 ft.; ·249 %.

3. 4 ft./sec.^{2} 4. 13·6 ft./sec.^{2} 5. $\frac{\Delta v}{\Delta t}$ ft./sec.^{2}, $\frac{dv}{dt}$.

6. (i) 32, 32, 32. (ii) -32, -32, -32. (iii) 18, 30, 102. (iv) 0, 0, 0 (ft./sec.^{2}).

7. ·54 ft.; ·36 ft./sec.

EXERCISES IX. p. 57.

1. 22·63 sq. ins. 2. ·14 %.

3. (i) 2π, (ii) 20π, (iii) 400π (sq. mm./sec.). 4. 40π sq. mm.

5. ·16 sq. ins. 6. (i) $\frac{1}{5\pi}=\cdot 064$ ins./sec. (ii) $\sqrt{\frac{1}{20\pi}}=\cdot 126$ ins./sec.

7. (i) $-\cdot 8$. (ii) $-\frac{1}{80}$. 8. (i) $\frac{1}{32}=\cdot 03125$ ins./sec. (ii) ·32 ins./sec.

EXERCISES X. p. 64.

1. $-6, -4, -2, 0, 2, 4, 6$. 2. 27, 3, 0, 75. 3. $-4, 0, 12, 28$.

4. 14; 14·03; about ·21 %. 5. (i) ·0025. (ii) ·00184. (iii) ·2. (iv) 2·5.

6. (i) $3-2x$. (ii) 8. (iii) $-\frac{2}{v^3}$.

EXERCISES XI. p. 63.

1. $\pm\frac{1}{\sqrt{24}}=\pm\cdot 204$, $\pm\frac{3}{4}$, $\pm\frac{4}{3}$. 2. -1; $-\frac{1}{4}$.

EXERCISES XII. p. 69.

1. $3kx^2$. 2. 108 ft./sec. 3. 60; $y=60x-80$.

4. 1·728 c. ins./sec. 5. $9\pi=28\cdot 27$ c. ft./sec. 6. 9·6 c. ins.

7. ·3 %.

EXERCISES XIII. p. 74.

1. $-\dfrac{1}{x^2}$. 2. $1-\dfrac{1}{x^2}$; 0; $\dfrac{7}{4}$. 3. $3x-4y+4=0$.

4. $2+2t$; 8 ft./sec.; 22 ft./sec.

5. (i) $-\dfrac{14}{x^3}$. (ii) $6x^2$. $\left.\begin{array}{r}14x+y-21=0\\ x-14y+97=0\end{array}\right\}$, $\left.\begin{array}{r}6x-y-1=0\\ x+6y-31=0\end{array}\right\}$.

6. $\dfrac{1}{2\sqrt{x}}$. 7. 0.

EXERCISES XIV. p. 79.

1. $5x^4$, $20x^{19}$, $\dfrac{1}{3}x^{-\frac{2}{3}}$, $-3x^{-4}$, $\dfrac{1}{4}x^{-\frac{3}{4}}$, $-\dfrac{4}{x^5}$, $1{\cdot}8x^{{\cdot}8}$, ${\cdot}4x^{-{\cdot}6}$, 1, 0, $\dfrac{3}{2}x^{\frac{1}{2}}$, $\dfrac{3}{4}x^{-\frac{1}{4}}$, $-\dfrac{1}{2}x^{-\frac{3}{2}}$, $-6x^{-7}$, $\dfrac{5}{3}x^{\frac{2}{3}}$, $-\dfrac{1}{2}x^{-\frac{3}{2}}$, $-\dfrac{2}{3}x^{-\frac{5}{3}}$.

2. $\dfrac{dy}{dx}=knx^{n-1}$. 3. $\dfrac{dy}{dx}=nx^{n-1}$.

EXERCISES XV. p. 80.

1. $15x^4$, $30x^{14}$, $-\dfrac{3}{x^2}$, $\dfrac{5}{2\sqrt{x}}$, $4x$, $-\dfrac{6}{x^3}$, $5{\cdot}64x^{{\cdot}41}$, $-{\cdot}273x^{-1{\cdot}091}$, x^2, $\dfrac{1}{10\sqrt{x}}$, $3x^2$, $\dfrac{7}{2\sqrt{x}}$.

2. $\dfrac{2}{3}t$. 3. $-\dfrac{700}{v^{2{\cdot}4}}$. 4. ak.

EXERCISES XVII. p. 84.

1. (i) $2+8x$, (ii) $\dfrac{6}{x^2}-\dfrac{2}{x^3}$. 2. $15t^2-t$.

EXERCISES XVIII. p. 85.

1. $3x^2-3$;

x	-2	-1	0	1	2
y	-2	2	0	-2	2
$\frac{dy}{dx}$	9	0	-3	0	9

2. $4x^3-12x^2+8x=4x(x-1)(x-2)$;

x	-1	0	1	2	3
y	6	-3	-2	-3	6
$\frac{dy}{dx}$	-24	0	0	0	24

3. $6x^2-18x+12=6(x-1)(x-2)$;

x	0	1	2	3
y	-3	2	1	6
$\frac{dy}{dx}$	12	0	0	12

4. $\left(\frac{3+\sqrt{3}}{3}, -\frac{2\sqrt{3}}{9}\right)$ and $\left(\frac{3-\sqrt{3}}{3}, \frac{2\sqrt{3}}{9}\right)$, or $(1{\cdot}577, -{\cdot}385)$ and $({\cdot}423, {\cdot}385)$.

EXERCISES XIX. p. 90.

1. $y=\frac{x^2}{4}$.

5. $3x-y-2=0$, $x+3y-4=0$; $\mathrm{T}\left(\frac{2}{3}, 0\right)$, $t(0, -2)$, $\mathrm{G}(4, 0)$, $g\left(0, \frac{4}{3}\right)$.

6. $96x-y-144=0$, $x+96y-4610=0$; $x+9y-6=0$, $9x-y-\frac{80}{3}=0$.

8. $\left.\begin{matrix}3x-y-4=0\\ x+3y-28=0\end{matrix}\right\}$, $\left.\begin{matrix}3x+y-4=0\\ x-3y-28=0\end{matrix}\right\}$.

9. $y=40x-57$, $x+40y=2523$. 10. $-\frac{5}{8}$, $\left(-\frac{16}{3}a, \frac{16}{3}a\right)$.

11. $y=7x-4$, $x+7y=72$, $\frac{100}{7}$ sq. units.

EXERCISES XX. p. 92.

1. $-3, -\frac{2}{3}, \frac{x+7}{2x+7}$. 2. $576\frac{4}{5}$, $3y^3+7y^2+5y+1+\frac{4}{y+1}$.

3. $\frac{\pi}{2}+1$, $1{\cdot}8415$, 0.

EXERCISES XXI. p. 97.

1. $2-\frac{1}{x^2}$; $\frac{2}{x^3}$; $4\frac{1}{2}$, $1\frac{3}{4}$, $\frac{1}{4}$. **2.** 35 ft./sec.; 22 ft./sec.^{2}

EXERCISES XXII. p. 101.

Letters refer to figures on p. 99.

1. (i) *a*. (ii) *c*. **2.** (i) *b*. (ii) *a*. (iii) *d*. **3.** (i) *b*. (ii) *a*.

MISCELLANEOUS EXAMPLES ON CHAPTER III. p. 102.

1. (i) $\frac{2}{3}x^{-\frac{1}{3}}$. (ii) $-\frac{2k}{v^3}=-\frac{2p}{v}$.

2. $\left.\begin{array}{r} x+4y-4=0 \\ 4x-y-\frac{15}{2}=0 \end{array}\right\}$; $\left.\begin{array}{r} x+25y-10=0 \\ 25x-y-\frac{624}{5}=0 \end{array}\right\}$; 2; 2.

3. $\left.\begin{array}{r} x+c^2y-2c=0 \\ c^2x-y-c^3+\frac{1}{c}=0 \end{array}\right\}$; 2. Area of triangle formed by the tangent at any point and the axes of co-ordinates is constant.

4. $\left.\begin{array}{r} 6x-y-9=0 \\ x+6y-57=0 \end{array}\right\}$; $(0, -9)$, $(0, 9\frac{1}{2})$; 18, $\frac{1}{2}$.

5.

t	5	6	7
s	207	301	413
v	85	103	121
a	18	18	18

6.

t	5	6	7
s	758	1341	2166
v	476	697	960
a	200	242	284

7. 14, 14·03, ·21 %. **8.** $y=144x-400$. $(0, 5)$, $(-1, 0)$, $(2, -27)$.

9. (i) $1-\frac{1}{n}$. (ii) $1-n$. (iii) n. (iv) $\frac{1}{n}$.

10. (i) $\frac{4\sqrt{3}}{100}=\cdot0693$ ins./sec. (ii) $\frac{\sqrt{3}}{100}=\cdot0173$ ins./sec.

11. 2·65. **12.** Average speed $=n$ times speed at end.

15. $(2\theta-8)\,a$, $-6\cdot704\times10^{-5}$, $2\cdot68\times10^{-4}$. **16.** 1·00896.

17. 0·172.

18.

x	-1	$-\frac{1}{4}$	0	1	2
y	4	$\frac{125}{128}$	1	0	13
$\frac{dy}{dx}$	-12		0	0	0	36

19. (i) 115 ft./sec. (ii) 123 ft./sec. (iii) 111 ft./sec.

20. $\left.\begin{matrix} y=12x-16 \\ x+12y=98 \end{matrix}\right\}$; $\frac{2}{3}$; 96. **22.** $\frac{3}{\pi}$ or ·955 ins./min. **23.** $(3\frac{1}{2}, 40\frac{1}{2})$.

24.

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	·30	·70	1·20	1·80	2·50	3·30	4·08	4·73	5·25	5·63	5·88	6·00
$\frac{dy}{dx}$	·25	·35	·45	·55	·65	·75	·85	·72	·58	·45	·32	·18	·05

25. $3x-2y=\frac{3}{8}$. **26.** 92·5 lbs. wt. **27.** 27 ft./sec.

28. 305 radians/sec. 120 radians/sec.^{2}

29.

$$\left(12-\frac{t}{1080}\right)\left(3-\frac{t}{720}\right)\ 375 \text{ lbs.,}$$

$$\left(12-\frac{t}{1080}\right)\left(3-\frac{t}{720}\right)\ 8250 \text{ lbs. ft./sec.}$$

$$\left(-\frac{7}{360}+\frac{t}{540\times 720}\right)\ 8250 \text{ poundals.}$$

(i) $-4·8$ lbs. wt., (ii) $-4·2$ lbs. wt.

30. $\frac{1}{9}$.

EXERCISES XXIII. p. 111.

1. $(1, -2)$ min., $(-1, 2)$ max. **3.** $\left(\frac{2}{3}, \frac{11}{27}\right)$ min., $\left(-\frac{3}{2}, \frac{83}{4}\right)$ max.

4. $(-1, 0)$ min., $(0, 1)$ max., $(1, 0)$ min. **5.** $(0, 1)$ min.

6. $(-1, 0)$ min., $(0, 5)$ max., $(2, -27)$ min.

7. $(-1, -2)$ max., $(1, 2)$ min. **8.** $\left(\frac{3}{4}, 3\frac{7}{8}\right)$ minimum;

EXERCISES XXIV. p. 118.

1. 1. **2.** $\sqrt[3]{\cdot 5}=\cdot 7937$. **3.** $\frac{4}{7}a$, $\frac{3}{7}a$. **4.** $2\frac{1}{2}$ ft. by $2\frac{1}{2}$ft.

7. Radius $\frac{2r}{3}$, height $\frac{h}{3}$. **8.** $\frac{100}{3}\sqrt{\frac{100}{3\pi}}=108\cdot 6$ c. ins.

9. $2r=h=\sqrt[3]{\frac{4V}{\pi}}$. **10.** 9·42 inches. **11.** $\sqrt[3]{150}=5\cdot 31$ knots.

12. $\frac{10}{\pi}=3\cdot 183$ acres. **13.** Total height = width $=\frac{60}{\pi+4}=8\cdot 40$ feet.

14. $\frac{2}{\sqrt{3}}=1\cdot 155$ ft., $2\sqrt{\frac{2}{3}}=1\cdot 633$ ft. **15.** $\frac{u^2\sin^2 a}{2g}$ ft.; $\frac{u\sin a}{g}$ secs.

16. $\frac{u^2\sin^2(a-\beta)}{2g\cos\beta}$ ft.; $\frac{u\sin(a-\beta)}{g\cos\beta}$ secs.

17. (i) 2 c. ft. (ii) $\frac{8}{\pi}=2\cdot 546$ c. ft. (iii) $\frac{8}{3\pi}=\cdot 849$ c. ft.

18. £7. 18*s*. 9*d*. [Ht. = $\frac{1}{2}$ side of base.] **19.** 4·8 ft., 3 ft., 6 ft.

20. rad. $=2\sqrt{6}$ ins., ht. $=4\sqrt{3}$ ins., vol. $=96\pi\sqrt{3}=522$ c. ins.

21. $\frac{1}{\sqrt{3}}$ (=·5774) of vol. of sphere; 1 : $\sqrt{2}$ or ·7071 : 1.

22. 2 radians = 114° 36′. **23.** $\frac{15}{11}$, $\frac{50}{11}$.

25. $r=10\sqrt[4]{\frac{4}{3\pi^2}}=6\cdot 06$, $l=10\sqrt[4]{\frac{12}{\pi^2}}=10\cdot 5$, volume = 330 c. ft.

26. 66·1°. **27.** 360·9 sq. ft. [$h=r\sqrt{2}$]. **28.** $\frac{16\pi}{9\sqrt{3}}=3\cdot 224$ c. ft

29. 5. **30.** $R=r$. **31.** 100.

32. $a=900$; $b=60{,}000$; $m=-404\cdot 8$; $n=733\cdot 6$; $V=358$.

34. $\sqrt{\frac{2aW}{w}}$. **35.** 4 of 6 ins. each and 2 of 2 feet each.

36. 45°. **37.** $13\frac{1}{3}$ feet; $\frac{al}{3(a-b)}$; $b=\frac{2}{3}a$.

38. $\sqrt{2bc}\,.\,\sin\frac{A}{2}$, where A is the least angle. **39.** $\frac{a}{2}$. **40.** $\frac{a}{2}$.

EXERCISES XXV. p. 134.

1. -71 (when $x=3$) min., 54 (when $x=-2$) max.; $\left(\frac{1}{2}, -8\frac{1}{2}\right)$.

2. $12\frac{3}{4}$ $\left(\text{when } x=\frac{3}{2}\right)$ Min., 44 (when $x=4$) Max.

3. $(1, -1)$. 4. $2+2\sqrt{\frac{5}{3}}=4\cdot58$ (when $x=\sqrt[4]{15}$).

5. (0, 0) max., $\left[\frac{12}{7}, -\frac{2}{7}\left(\frac{12}{7}\right)^6\right]$ or $[1\cdot71, -7\cdot25]$ min. 1 point of inflexion $\left[\frac{10}{7}, -\left(\frac{4}{7}\right)\left(\frac{10}{7}\right)^6\right]$ or $[1\cdot43, -4\cdot86]$.

7. A minimum point.

8. $(3, -47)$ Min., $(-1, 17)$ Max., $(1, -15)$ point of inflexion.

EXERCISES XXVI. p. 138.

2. 45·24 sq. ins. 3. 362 c. ins. 4. 197 ft., 9·8 ft.

5. (i) 19·6 c. ft. (ii) 7·07 c. ft. 6. $\pi r^2 \cot a \,.\, \Delta r$.

7. (i) 46·5 sq. ins. (ii) 430 c. ins. 8. $R_0\,(a+2bt)$, ·0064.

EXERCISES XXVII. p. 139.

1. 1·5 %. 3. Decrease 1 %. 10. 6·08 ft./sec., ·025.

11. 2·214, ·0115. 12. 1·1 % decrease.

EXERCISES XXVIII. p. 143.

2. 37·2088. 3. 96·244. 4. 24·24.

5. (i) 1. (ii) 1. 1·00007525, 1·00007475.

6. (i) -3. (ii) -2. $-2\cdot99939197$, $-2\cdot00000397$.

EXERCISES XXX. p. 146.

[Add an arbitrary constant in each case.]

1. (i) x^3. (ii) $\frac{3}{4}x^4$. (iii) $-\frac{2}{x}$. (iv) $7x$. (v) $10x^{\frac{1}{2}}$. (vi) $\frac{2}{3(n+1)}x^{n+1}$.

(vii) $-\frac{1}{4x^2}$. (viii) $\frac{3}{4}x^{\frac{4}{3}}$. (ix) $\frac{4}{3}x^{\frac{3}{4}}$. (x) $\frac{12}{55}x^{\frac{11}{4}}$. (xi) $\frac{1}{2}x^4+\frac{3}{2}x^2$.

(xii) $-\frac{3}{x}+4x$. (xiii) $\frac{7}{3}x^3+\frac{3}{2}x^2-\frac{1}{x^2}$. (xiv) $3x^4+4x^{\frac{3}{2}}$.

2. (i) $\frac{x^3}{3}$. (ii) $x^3-\frac{1}{2}x^2+x$. (iii) $-\frac{5}{x}+3x$.

EXERCISES XXXI. p. 147.

[a and b are arbitrary constants.]

1. (i) $\frac{x^4}{4}+ax+b$. (ii) $\frac{5}{2x}+ax+b$. (iii) $\frac{7}{2}x^2+ax+b$. (iv) $\frac{x^3}{9}+\frac{5}{2}x^2+ax+b$.

(v) $\frac{1}{9x^2}+ax+b$. (vi) $\frac{4}{15}x^{\frac{5}{2}}+ax+b$.

2. $\frac{x^3}{2}-4x^2+ax+b$. **3.** $\frac{5}{2}x^2+ax+b$. **4.** $\frac{x^5}{10}-\frac{x^3}{2}+\frac{x^2}{2}+ax+b$

EXERCISES XXXII. p. 148.

1. $y=\frac{x^3}{3}+x-9$. **2.** $p=\frac{v^2}{2}-\frac{1}{v}+\frac{7}{2}$. **3.** $y=\frac{2}{3}x^{\frac{3}{2}}+\frac{5}{3}$.

4. $A=\frac{5}{2}y^2+4y-\frac{1}{2}$. **5.** $y=\frac{x^3}{3}+\frac{3x^2}{2}+2x-\frac{33}{2}$.

6. $s=\frac{4}{3}t^3-6t^2+9t-\frac{29}{3}$. **7.** $V=\frac{4}{3}\pi r^3$. **8.** $y=\frac{3}{2}x^{\frac{2}{3}}-\frac{11}{2}$.

9. $s=ut+\frac{1}{2}at^2$. **10.** $y=\frac{x^3}{2}+4x-1$. **11.** $y=16x^2+5x$.

EXERCISES XXXIII. p. 150.

1. $y=5+3x-2x^2$. **2.** (i) $y=3x-2x^2$. (ii) $y=28+3x-2x^2$.

4. $y=x^2+x+3$; $y=3x+2$.

5. $y=\frac{2}{3}x^3+\frac{3}{2}x^2-7x+\frac{41}{6}$; $x-2y+3=0$.

6. (i) $\left(\frac{5}{3}, -\frac{25}{6}\right)$. (ii) $\left(\frac{5}{3}, -\frac{92}{3}\right)$.

7. $y=-7+7x-2x^2+\frac{1}{3}x^3$; $(0, -7)$; $8° 8'$.

EXERCISES XXXIV. p. 152.

1. $(60t-16t^2)$ ft. **2.** 186 ft./sec.; 404 ft.

3. $s=\frac{1}{2}t^3+t^2+t+5$. (i) 5 ft., 1 ft./sec. (ii) $97\frac{1}{2}$ ft., $48\frac{1}{2}$ ft./sec.

5. $p=\frac{500}{v^{1 \cdot 4}}+11 \cdot 41$. **6.** $y=x^2+x+3$.

MISCELLANEOUS EXAMPLES ON CHAPTERS I—VI. p. 153.

A.

1. $\frac{dy}{dx}=9x^2+2$. $y=11x-5$. **2.** $6\frac{2}{3}$ ft. **3.** 1 ft., $\frac{1}{2}$ ft.

4. (i) (*a*) $\cdot 08\pi$ or $\cdot 2513$ sq. ins./sec. (i) (*b*) $\cdot 96\pi$ or $3 \cdot 016$ sq. ins./sec.
(ii) (*a*) $\cdot 04\pi$ or $\cdot 1257$ c. ins./sec. (ii) (*b*) $5 \cdot 76\pi$ or $18 \cdot 096$ c. ins./sec.

5. Height = diameter $=2\sqrt[3]{\frac{150}{\pi}}=7 \cdot 26$ cms. Surface = 248 sq. cms.

B.

1. (i) $-\frac{1}{x^2}$. (ii) $x+25y=10$, $25x-y=124\frac{4}{5}$.

2. (i) $\frac{5}{2\sqrt{x}}-x^{-\frac{2}{3}}$. (ii) $\frac{\pi}{3}(2Rx-3x^2)$. **3.** $\frac{5}{6}\sqrt[3]{\frac{1}{6\pi}}=\cdot 313$ ft./min.

4. Greatest when radius of circle $=\frac{50}{\pi}$ ft. $=15\cdot 9$ ft. and side of square $=0$ (not a true max.). Area $=796$ sq. ft.
Least when side of square = diameter of circle $=\frac{100}{\pi+8}$ ft. $=8\cdot 97$ ft. (true min.). Area $=\frac{2500}{\pi+8}=224$ sq. ft.

5. $y=\frac{3}{2}x^2$.

C.

1. $\frac{dy}{dx}=6x$; $2\frac{1}{12}$.

2. Max. 142 when $x=-2$; Min. -2055 when $x=11$.

3. (i) 150 ft./sec. (ii) $58\cdot 4$ ft./sec. **4.** $(3\frac{1}{3}, 40\frac{1}{2})$, $x=3\frac{1}{3}$.

5. $f(x)=x^3-3x^2+x+c$; -2.

D.

1. 80 ft./sec., -36 ft./sec., $87\cdot 72$ ft./sec. at angle $65°\,46'$ with downward vertical.

2. Max. $\frac{160+56\sqrt{28}}{27}$ $(16\cdot 90)$ when $x=\frac{1-\sqrt{28}}{3}$ $(-1\cdot 431)$.
Min. $\frac{160-56\sqrt{28}}{27}$ $(-5\cdot 05)$ when $x=\frac{1+\sqrt{28}}{3}$ $(2\cdot 097)$.

3. $24x-5y+72=0$, $8x+5y-8=0$, $12x-5y-36=0$; $9x+5y-9=0$; $\left(\frac{1}{3}, \frac{32}{27}\right)$.

4. $\frac{2\cdot 5}{\sqrt{\pi}}=1\cdot 41$ c. ins. **5.** $y=\frac{4}{3}x^3+3x-\frac{10}{3}$.

E.

1. $\cdot 6\,\%$. **2.** 20 ft. by 20 ft. by 10 ft.

3. (i) (64, 32). (ii) $x-2y+16=0$.
(iii) $\left[64\left(1\mp\frac{1}{\sqrt{3}}\right), \frac{64}{3}\right]$ or $(27\cdot 146, 21\frac{1}{3})$ and $(100\cdot 954, 21\frac{1}{3})$.

4.

x	-1	$-\frac{1}{4}$	0	1	2
y	4	$\frac{125}{128}$	1	0	13

$\frac{dy}{dx}=2x(4x+1)(x-1)$. -12; 36; $-\frac{1}{4}$, 0, 1.

5. (i) $46\frac{1}{3}$ ft./sec. (ii) 49 ft./sec. (iii) 45 ft./sec.

F.

1. 3·91° C. **2.** $\left(\frac{1}{3}, \pm\cdot385\right)$; $y=x-1$; $y=-x+1$.

4. $1\cdot09\times10^4$ ft. lbs. wt.; 3740 lbs. ft./sec.; 60 lbs. wt. **5.** 18·18.

G.

1. 1080 ft. **3.** Lengths of pieces are $\frac{4a}{\pi+4}$, $\frac{a\pi}{\pi+4}$ inches, or ·56a, ·44a.

5. 13·8.

H.

1. $(x-2)^2(4x+1)$; $6(x-2)(2x-1)$; $6(4x-5)$.

(i) $x=-1$ or 2. (ii) $x=-\frac{1}{4}$ or 2. (iii) $x=\frac{1}{2}$ or 2. (iv) $x=\frac{5}{4}$.

Min. pt. on $y=f(x)$ $\left(-\frac{1}{4}, -\frac{2187}{256}\right)$.

$\begin{cases}\text{Max. pt. on } y=f'(x)\left(\frac{1}{2}, \frac{27}{4}\right). \\ \text{Min. pt. on } y=f'(x)\ (2, 0).\end{cases}$

Min. pt. on $y=f''(x)$ $\left(\frac{5}{4}, -\frac{27}{4}\right)$.

Pts. of infl. on $y=f(x)$ (2, 0), $\left(\frac{1}{2}, -\frac{81}{16}\right)$.

Pt. of infl. on $y=f'(x)$ $\left(\frac{5}{4}, \frac{27}{8}\right)$.

x	-1	0	1	2	3
$f(x)$	0	-8	-2	0	4
$f'(x)$	-27	4	5	0	13
$f''(x)$	54	12	-6	0	30
$f'''(x)$	-54	-30	-6	18	42

4. $\frac{r}{3}$ below centre. **5.** (3·99, 2·26); (1·05, 4·46).

I.

1. 20 ins./sec. **3.** (i) $(3x+2y)$ ins. (ii) $\left(xy+\frac{x^2\sqrt{3}}{4}\right)$ sq. ins.

(iii) $\frac{198+33\sqrt{3}}{4}=63\cdot79$ sq. ins.

4. (i) $7\frac{1}{2}$ ft./sec. (ii) $12\frac{1}{2}$ ft./sec. **5.** $3\cdot32\times10^4$ cms./sec.

J.

1. 425 ft. 2. 4·5 %.

3. $\dfrac{dy}{dx}$ is + from $x=-2$ to $x=2$ and − when $x>2$.

4. $\dfrac{y}{\dfrac{dy}{dx}}=3x$. 5. $1\frac{1}{2}$ m./hr., $4\frac{1}{2}$ m./hr., $13\frac{1}{3}$ hours.

EXERCISES XXXV. p. 168.

1. $97\frac{1}{2}$; ·975 sq. ins.; 4·875 sq. ins. 2. 4; 6 sq. ins.

3. (i) 285, 385. (ii) 328·35, 338·35.
(iii) $\frac{5}{3}(200-30h+h^2)$, $\frac{5}{3}(200+30h+h^2)$. (iv) $333\frac{1}{3}$.

4. $20\frac{1}{4}$. [Inside rects. $\left\{\dfrac{3(3-h)}{2}\right\}^2$, outside $\left\{\dfrac{3(3+h)}{2}\right\}^2$.]

EXERCISES XXXVI. p. 171.

1. 108. 2. $121\frac{1}{2}$. 3. $13\frac{1}{2}$. 4. $28\frac{3}{4}$.
5. $97\frac{1}{2}$. 6. 4. 7. $23\frac{3}{8}$.

EXERCISES XXXVII. p. 172.

1. $905\frac{3}{5}$. 2. $4\frac{2}{3}$. 3. $152\frac{1}{4}$. 4. $274\frac{3}{5}$.
5. 3696. 6. $48\frac{3}{4}$. 7. $\dfrac{b^4-a^4}{4}$.

EXERCISES XXXVIII. p. 177.

(i) $A=\frac{5}{4}x^4+2x$. (ii) $A=\frac{5}{4}x^4+2x-107\frac{1}{4}$.

(iii) $A=\frac{5}{4}x^4+2x-16$. (iv) $11728\frac{3}{4}$.

EXERCISES XXXIX. p. 181.

[In 1—9 an arbitrary constant should be added.]

1. $\frac{2}{3}x^3$. **2.** $2x^{\frac{3}{2}}-\frac{1}{2x^2}$. **3.** $\frac{a}{3}x^3+\frac{b}{2}x^2+cx$. **4.** $\frac{1}{2}x^{\frac{2}{3}}$.

5. $\frac{5}{2}t^2+2t^3-\frac{3}{5}t^{\frac{5}{3}}$. **6.** $-\frac{5}{2v^{\cdot 4}}$. **7.** $7x$.

8. $\frac{3}{4}x^4-\frac{5}{3}x^3+x^2-3x-\frac{1}{3x}$. **9.** $\frac{1}{2}x^2-\frac{1}{x}$. **10.** $223\frac{1}{3}$.

11. 4·032. **12.** $\frac{2a}{3}+2c$. **13.** $\frac{3}{2}$. **14.** $1164\frac{4}{5}$.

15. ·0695. **16.** 56. **17.** $614\frac{1}{36}$. **18.** 2.

19. (i) 278. (ii) 225·9875. **20.** (i) 56. (ii) 56. **21.** 2·431.

EXERCISES XL. p. 183.

1. $2\frac{2}{3}$. (ii) $5\frac{1}{3}$. **2.** (i) 4. (ii) 12. **3.** (i) $\frac{h^3}{3a}$. (ii) $\frac{2h^3}{3a}$.

6. $211\frac{1}{2}$. **7.** $\frac{2ap^3}{3}+2cp$. **8.** $p=l,\ q=\frac{k-h}{2a},\ r=\frac{h+k-2l}{2a^2}$.

EXERCISES XLI. p. 187.

1. 5180. **2.** (i) 6055. (ii) 5180. **3.** (i) 5215. (ii) 5180.

4. (1) 3355. (2) (i) 4046·41. (2) (ii) 3381·04.
(3) (i) 3382·91. (3) (ii) 3355·04.

EXERCISES XLII. p. 190.

2. 1·4. **3.** $\frac{2}{3}$. **4.** 14. **5.** $\frac{8}{15}$.

6. $a=0,\ b=-8,\ c=6$. 4.

7. $a=h(h^2-k^2),\ b=k^2-3h^2,\ c=3h$. Each area $=2hk(h^2+k^2)$.

9. $2pk+\frac{2rk^3}{3}$. **10.** $\frac{a}{3}(h_1+4h_2+h_3)$. **11.** $4\frac{1}{2}$.

12. $8\frac{3}{16}$, $1\frac{1}{3}$. **13.** $\frac{16}{3}a^2$. **14.** $\frac{16}{5}\sqrt{2}=4\cdot 53$. **15.** $5\frac{5}{24}$.

EXERCISES XLIII. p. 193.

2. ·69315. **3.** ·1419, $-$·2027.

EXERCISES XLV. p. 200.

1. 52, 58, 49. **2.** 32, 32, 32.

4. (i) 145. (ii) $\dfrac{(a+b)(a^2+b^2)}{4}$. **5.** $1\frac{1}{3}$. **6.** 8.

EXERCISES XLVI. p. 202.

1. $18\frac{1}{3}$, 18·4. **2.** $\frac{1}{4}$. **3.** ·69315, ·6989. **4.** $170\frac{11}{12}$.

EXERCISES XLVII. p. 204.

1. 0. **2.** (i) $6\frac{3}{4}$. (ii) 0. **3.** $3\frac{1}{6}$, 2·75. **4.** $4\frac{1}{6}$.

5. $\dfrac{2}{3}a^2$. **6.** $11\frac{1}{3}$.

EXERCISES XLVIII. p. 206.

(i) $112\frac{1}{2}$, 112·325, ·155 %. (ii) $71\frac{1}{2}$, 71·465, ·05 %.

EXERCISES XLIX. p. 208.

1. 61 ft. **2.** 5131 ft.

EXERCISES L. p. 227.

2. $304\frac{1}{2}$ ft. **3.** $1117\frac{1}{3}$ ft./sec. $193\frac{1}{3}$ ft./sec. $v=\dfrac{3t^2}{2}+\dfrac{7t^3}{3}+20$, 2544 ft.

4. $1·19\times10^4$ ft. lbs. wt. **5.** 560 ft. lbs. wt. **6.** 9000 lbs. wt.

7. 750 lbs. wt. **8.** 375 lbs. wt. **9.** $\dfrac{k(b-a)}{l}\left[\dfrac{b+a}{2}-l\right]$ ft. lbs. wt.

10. 2,700,000 ft. lbs. wt. 6·71 knots. **11.** 192 ft./sec. $346\frac{2}{3}$ ft.

12. $\dfrac{m\omega^2r^2}{2g}$ ft. lbs. wt., ωr ft./sec. **13.** $10{,}560{,}000m$ ft. lbs. wt.

14. 60 lbs.

EXERCISES LI. p. 232.

1. $120\pi = 376{\cdot}99$ c. ins. **2.** $\frac{1}{3}\pi r^2 h$ c. ins. **3.** $\frac{64\pi}{3} = 67{\cdot}02$ c. ins.

4. $\frac{\pi h^2}{3}(3r - h)$. **5.** (i) $261\pi = 820$ c. ins. (ii) $\frac{1967}{3}\pi = 2060$ c. ins.

6. (i) $16\pi = 50{\cdot}27$. (ii) $24\pi = 75{\cdot}40$. **7.** $\frac{32\pi}{3} = 33{\cdot}51$.

8. $\frac{\pi a k^2}{2}$. **9.** $2\frac{38}{45} = 2{\cdot}84$; $\frac{16}{27}\pi = 1{\cdot}862$.

10. $y = 5 - \frac{1}{2}x^2$. $\frac{1148}{15}\pi = 240{\cdot}4$.

11. (i) $120\pi = 376{\cdot}99$. (ii) $60\pi = 188{\cdot}5$.

EXERCISES LII. p. 237.

1. (*a*) $666\frac{2}{3}$ lb. ft.^{2} (*b*) $246\frac{2}{3}$ lb. ft.^{2} **2.** $M\frac{b^2}{3}$. **3.** $M\frac{b^2}{12}$.

4. (i) $M\frac{a^2+b^2}{3}$. (ii) $M\frac{a^2+b^2}{12}$. **5.** $\frac{2}{5}Ma^2$.

6. (i) $\sqrt{\frac{ak}{5}}$. (ii) $\sqrt{\frac{ak}{3}}$. **7.** $M\left(\frac{a^2}{12}+p^2\right)$.

8. (i) $\frac{r}{\sqrt{2}}$. (ii) $r\sqrt{\frac{3}{10}}$. **9.** $\frac{3M}{2}$. **10.** $2{\cdot}025$, $1{\cdot}62$, $1{\cdot}35$ ft.

11. $M\frac{r^2}{2}$. **12.** $M\frac{R^2+r^2}{2}$. **13.** $85{\cdot}7$ ft. lbs. wt.

14. (i) $\frac{2}{5}Mb^2$. (ii) $\frac{2}{5}Ma^2$. **15.** $6{\cdot}34$M. $\left(M\frac{2343}{370}\right)$.

EXERCISES LIII. p. 244.

1. $\left(\frac{2}{3}a,\ \frac{1}{3}ka\right)$. **2.** $\frac{h}{3}$. **3.** $\left(0,\ \frac{3}{5}k\right)$. **4.** $\left(\frac{3}{4}h,\ 0\right)$.

5. $(7{\cdot}71,\ 85{\cdot}34)$. **6.** $\left(0,\ \frac{2p+a}{3(p+a)}q\right)$.

7. $\frac{4a}{3\pi}$ from centre, on a radius perpendicular to diameter.

8. $\frac{3}{8}a$ from centre, on a radius perpendicular to base.

EXERCISES LIV. p. 248.

1. $\frac{a\sqrt{5}}{2} = 1{\cdot}118a$. 2. $M\left(\frac{a^2}{2}+c^2\right)$. 3. $M\left(\frac{a^2}{4}+c^2\right)$.

4. $2Ma^2$. 5. $M\left(\frac{a^2}{4}+c^2\right)$. 6. $M\left(\frac{h^2}{12}+\frac{r^2}{4}\right)$.

7. $\sqrt{\frac{3}{5}\left(h^2+\frac{r^2}{4}\right)}$.

EXERCISES LV. p. 249.

1. On vertical line of symmetry (i) $\frac{2b}{3}$, (ii) $\frac{3c+2b}{2c+b}\cdot\frac{b}{3}$ below the top side.

2. On the median drawn to the base a (i) $\frac{3}{4}h$ below, (ii) $\frac{1}{2}h$ above, (iii) $\frac{4c+3h}{3c+2h}\cdot\frac{h}{2}$ below, (iv) $\frac{2c+h}{3c+h}\cdot\frac{h}{2}$ above the vertex.

3. On the line joining the mid-points of the parallel sides (i) $\frac{a+3b}{a+2b}\cdot\frac{h}{2}$, (ii) $\frac{h(a+3b)+c(2a+4b)}{h(a+2b)+c(3a+3b)}\cdot\frac{h}{2}$ below the side a.

EXERCISES LVI. p. 253.

1. $\frac{2}{3}\pi a^2$. [Area of base of cylinder, same height and volume.]

2. $\frac{\pi r^2}{3}$.

4. $u+\frac{1}{2}a(t_1+t_2)+\frac{1}{3}b(t_1^2+t_1t_2+t_2^2)$; $u+\frac{1}{2}a(t_1+t_2)+\frac{1}{2}b(t_1^2+t_2^2)$.

5. 420 lbs. wt./sq. ft. 6. (i) 7. (ii) $7\frac{3}{7}$.

MISCELLANEOUS EXAMPLES ON CHAPTERS VII AND VIII. p. 254.

A.

1. $\dfrac{20\sqrt{10}+52}{3}$ (38·41); $\dfrac{20\sqrt{10}-52}{3}$ (3·75). **2.** 300π. **3.** $95\frac{1}{3}$.

5. $\dfrac{3}{4}$ of the way from A down the median.

$\dfrac{5}{8}$ of the way from A up the median.

B.

1. $\dfrac{2}{3}x^3-\dfrac{3}{2}x^2+x+c$; $\dfrac{2}{5}x^{\frac{5}{2}}+c$; $-\dfrac{1}{x}+c$.

2. $52\frac{1}{12}$; $\dfrac{500}{27}\left(18\frac{14}{27}\right)$; $\dfrac{2500}{27}\left(92\frac{16}{27}\right)$. **3.** $\dfrac{8\pi}{3}$; $\dfrac{16\pi}{3}$.

4. -5, 0, 4; 42·04, -28·56; 45, -20, 36; $135\frac{5}{12}$, $74\frac{3}{4}$.

5. $\mathrm{M}\left(c^2+\dfrac{a^2}{4}\right)$; $\dfrac{3\mathrm{M}}{80}(h^2+4r^2)$.

C.

1. $y=4x$; $y+8x=16$; $\dfrac{16\sqrt{3}}{9}=3{\cdot}079$; 4. **2.** $\dfrac{2\pi h}{3}(3a+ch^2)$.

3. $\mathrm{M}\dfrac{h^2}{6}$ (where h is the altitude from A) $=\dfrac{2\mathrm{M}}{3a^2}s(s-a)(s-b)(s-c)$.

4. 139 ft. lbs. wt.; 46·5 lbs. wt./sq. ft. **5.** $\dfrac{7\sqrt{2}}{12}.a$.

D.

1. 30; 4; $26\frac{1}{3}$. **2.** (i) $20\frac{1}{4}$. (ii) $20\frac{1}{4}$. **3.** 10 inches.

4. $3\frac{3}{4}$ m. **5.** $8{\cdot}713\times10^8$ gm. cm.^{2}; $1{\cdot}075\times10^9$ ergs.

E.

2. $135\pi = 424 \cdot 1$ c. ft. $\sqrt{22 \cdot 5} = 4 \cdot 743$ ft.

3. $\tan^{-1}\dfrac{3}{41} = 4^\circ\ 11'$; $0 \cdot 122$; $(1 \cdot 61,\ 1 \cdot 12)$.

4. $\dfrac{m}{2\text{R}}(2a\text{R} - a^2)$ ft. lbs. wt.; $\dfrac{ma\text{R}}{(\text{R}+a)}$ ft. lbs. wt.; 35 : 32.

F.

1. (i) $\dfrac{6}{5}x^{\frac{5}{2}} + c$. (ii) $37\frac{1}{3}$. **2.** $\dfrac{7}{3},\ -\dfrac{21}{2},\ \dfrac{73}{6},\ 0$; $7\frac{1}{2}$.

3. $\cdot 4514,\ 2 \cdot 374$; 1160. **4.** $\dfrac{\text{M}}{3}(l^3 - 3al + 3a^2)$; $a = \dfrac{l}{2}$.

G.

1. $\dfrac{1}{4}$, $12\frac{1}{2}$ sq. ins.; $\dfrac{1}{4}$, $2\frac{1}{2}$ ins.; $\dfrac{5}{36}$, $6\frac{17}{18}$ sq. ins. **5.** $\dfrac{287}{54}\pi = 16 \cdot 7$ c. ft.

H.

1. $7\frac{13}{20}$, $\frac{53}{60}$, $1\frac{1}{3}$; $\frac{11}{15}$.

2. (i) 27. (ii) $\left(\dfrac{1}{2},\ -\dfrac{27}{5}\right)$. (iii) $291 \cdot 6\pi = 916$. (iv) $\left(\dfrac{1}{2},\ 0\right)$.

3. $y = \dfrac{5}{2} - \dfrac{1}{18}x^2$; $\dfrac{164}{5}\pi = 103$ c. ft. **4.** $r\sqrt{\dfrac{2}{5}} = \cdot 6325r$.

5. 360 ft. lbs. wt.

I.

3. $\sqrt{\dfrac{17\pi + 72}{16(\pi + 6)}} = \cdot 928$ ins. **5.** $\dfrac{8\sqrt{3}}{3} = 4 \cdot 62$.

J.

1. $\dfrac{4096}{165}\pi = 78 \cdot 0$. **2.** $\dfrac{12}{\pi} = 3 \cdot 82$ ins./min. **3.** 439 ft. lbs. wt.

4. $\dfrac{1}{5}$; $\dfrac{5\pi}{28} = \cdot 562$. **5.** $\dfrac{mra}{r+a}$ ft. lbs. wt. $\left[= \dfrac{mr}{1 + \dfrac{r}{a}} \right]$.

EXERCISES LVII. p. 268.

3. 2, $-1{\cdot}444$, $-3{\cdot}560$, $0{\cdot}232$, $-3{\cdot}598$, 0, $\sqrt{13}=3{\cdot}606$, $-\sqrt{13}=-3{\cdot}606$.
$y=2x-1$, $y+{\cdot}144x=4{\cdot}748$, $y+3{\cdot}598x=7{\cdot}77$.
$y=3{\cdot}606x+3{\cdot}545$, $y=3{\cdot}606$.

5. (i) $-5\sin 5x$. (ii) $6\cos 2x$. (iii) $\cos 4x+\sin 3x$.
(iv) $n(a\cos nx-b\sin nx)$.

6. $a\cos\omega t$, $a\sin\omega t$, $-a\omega\sin\omega t$, $-a\omega^2\cos\omega t$, $a\omega\cos\omega t$, $-a\omega^2\sin\omega t$.

9. (i) $-\sin x$. (ii) $-\cos x$. **10.** (i) $-an^2\cos nx$. (ii) $-an^2\sin nx$.

11. (i) $\sqrt{13}$, $-\sqrt{13}$. (ii) $\sqrt{41}$, $-\sqrt{41}$.

12. (i) $-\cos x+c$. (ii) $\sin x+c$. **13.** 2, $\frac{4}{\pi}=1{\cdot}273$ sq. ins.

14. (i) $0{\cdot}4598$. (ii) $0{\cdot}8415$. (iii) 1. (iv) 1.

15. (i) 0. (ii) 0. (iii) 0. (iv) -2.

16. (i) $-\frac{1}{n}\cos nx+c$. (ii) $\frac{1}{n}\sin nx+c$.

18. [An arbitrary constant to be added in each case except those marked *.]

(i) $-\frac{1}{2}\cos 2x$. (ii) $\frac{1}{2}\sin 2x$. (iii) $-\frac{3}{2}\cos 2x+\frac{4}{3}\sin 3x$.

(iv) $x+\frac{1}{2}\sin 2x$. (v) 0^*. (vi) 0^*. (vii) $-\frac{1}{4}\cos 2x$.

(viii) $\frac{x}{2}-\frac{\sin 2x}{4}$. (ix) $\frac{x}{2}+\frac{\sin 2x}{4}$. (x) $\frac{\pi}{2}^*$. (xi) $\frac{\pi}{2}^*$.

19. $\left(\frac{\pi}{2}, \frac{\pi}{8}\right)$ or $(1{\cdot}5708,\ 0{\cdot}3927)$. **20.** π.

21. [Add an arbitrary constant.]

(i) $-\frac{1}{2}\left(\frac{\cos 5x}{5}+\cos x\right)$. (ii) $-\frac{1}{2}\left(\frac{\cos 5x}{5}-\cos x\right)$.

(iii) $\frac{1}{2}\left(\frac{\sin 5x}{5}+\sin x\right)$. (iv) $\frac{1}{2}\left(-\frac{\sin 5x}{5}+\sin x\right)$.

(v) $-\frac{1}{2}\left(\frac{\cos(p+q)x}{p+q}+\frac{\cos(p-q)x}{p-q}\right)$.

(vi) $\frac{1}{2}\left(\frac{\sin(p+q)x}{p+q}+\frac{\sin(p-q)x}{p-q}\right)$.

(vii) $\frac{1}{2}\left(-\frac{\sin(p+q)x}{p+q}+\frac{\sin(p-q)x}{p-q}\right)$.

22. ·5150. **23.** ·96 sq. ins. **24.** 6.

25. ·791. **26.** $-1{\cdot}745$ ft./sec., $-4{\cdot}685$ ft./sec.^{2}

27. (i) $\frac{2r}{\pi} = {\cdot}6366r$. (ii) $\frac{\pi r}{4} = {\cdot}7854r$.

EXERCISES LVIII. p. 274.

4. (i) $-n \operatorname{cosec}^2 nx$. (ii) $+n \sec nx \tan nx$. (iii) $-n \operatorname{cosec} nx \cot nx$.

6. An arbitrary constant to be added in every case.

(i) $\tan x$. (ii) $-\cot x$. (iii) $\sec x$. (iv) $-\operatorname{cosec} x$. (v) $\tan x$.

(vi) $-\cot x$. (vii) $\sec x$. (viii) $-\operatorname{cosec} x$. (ix) $\frac{1}{3}\tan 3x$.

(x) $\tan x - x$. (xi) $\frac{1}{3}\tan 3x - x$.

8. 2500, 1250, 150·75, 1·523, 0·381, 0·095 ft./sec.;
$\frac{1}{3}$, $\frac{2}{3}$, 5·53, 547, 2189, 8754 ft./sec.

EXERCISES LIX. p. 277.

1. 15·535 ft. **2.** ·1396 ft./sec. **3.** 35·8 mins./sec. or $\frac{1}{96}$ radians/sec.

4. 9 ft./sec.; 17° 11′ per sec. or $\frac{3}{10}$ radian/sec.

EXERCISES LX. p. 279.

2. (i) $-(2x+3)\sin x + 2\cos x$. (ii) $x^3 \sec^2 x + 3x^2 \tan x$.

EXERCISES LXI. p. 281.

1. $x\cos x + \sin x$. **2.** $x^2 \sec^2 x + 2x \tan x$. **3.** $18x^2 + 26x + 8$.

4. $2\cos 2x$. **5.** $\frac{7}{10} x^{-\frac{3}{10}}$. **6.** $\sec x \tan x$.

7. $4x^5 \cos x + 20x^4 \sin x$. **8.** $-15x^2 \sin 3x + 10x \cos 3x$.

9. $-2\cos x \sin x$. **10.** (i) $-10\cos 5x \sin 5x$. (ii) $10 \sin 5x \cos 5x$.

11. $\frac{\cos x}{2\sqrt{x}} - \sqrt{x}\sin x$. **12.** $\frac{\cos x}{x^2} - \frac{2\sin x}{x^3}$.

EXERCISES LXII. p. 282.

1. $4x^3+12x^2+6x$. 2. $1-\frac{2}{x^2}$. 3. $x\sin 2x+x^2\cos 2x$.

4. $3(4x+3)(2x^2+3x+1)^2$. 5. (i) $2u.\frac{du}{dx}$. (ii) $3u^2.\frac{du}{dx}$. (iii) $4u^3.\frac{du}{dx}$.

EXERCISES LXIII. p. 285.

1. $\frac{1}{(x+2)^2}$. 2. $\frac{1-x^2}{(1+x^2)^2}$. 3. $\frac{x\cos x-\sin x}{x^2}$.

4. $\frac{\sin x-x\cos x}{\sin^2 x}$. 5. $\frac{-8x^2+2x+11}{(x^2-5x+2)^2}$. 6. $-\frac{2x+3}{(x^2+3x+1)^2}$.

7. $-\frac{2\cos x}{(1+\sin x)^2}$. 8. $\frac{\sin x}{\cos^2 x}$. 9. $-\frac{1}{u^2}.\frac{du}{dx}$.

EXERCISES LXIV. p. 286.

1. $4x^9$. 2. $3x^4$. 3. $\frac{1}{2\sqrt{\sin x}}$. 4. $\sec^2 4x$.

5. $6\sec^5 x$. 6. $-\frac{1}{(2x+3)^2}$. 7. $2\cos(4x+7)$. 8. 1.

9. $\frac{1}{2\sqrt{x^2+x+1}}$. 10. $\frac{5}{2}(2\sin x+3\cos x)^{\frac{3}{2}}$.

EXERCISES LXV. p. 288.

1. $-5\sin 5x$. 2. $3\sec^2 3x$. 3. $3\sin^2 x\cos x$.

4. $2x\cos(x^2+3)$. 5. $\frac{1}{2\sqrt{x}}\cos\sqrt{x}$. 6. $14(2x+3)^6$.

7. $4(6x+5)(3x^2+5x+1)^3$. 8. $21\sin^2(7x+5)\cos(7x+5)$.

EXERCISES LXVI. p. 290.

1. $\frac{\cos x}{2\sqrt{\sin x}}$. 2. $\frac{1}{2\sqrt{x}}\cos\sqrt{x}$. 3. $4\sin^3 x\cos x$.

4. $4\cos 4x$. 5. $4x^3\cos(x^4)$. 6. $n\sin^{n-1}x\cos x$.

7. $n\cos nx$. **8.** $n\cos x$. **9.** $-n\cos^{n-1}x\sin x$.

10. $-n\sin nx$. **11.** $4\cos(4x+5)$.

12. $2(3\sin x+4\cos x)(3\cos x-4\sin x)$. **13.** $2\tan x\sec^2 x$.

14. $2\sec^2 x\tan x$. **15.** $n(a\sin x+b\cos x)^{n-1}(a\cos x-b\sin x)$.

16. $n\tan^{n-1}x\sec^2 x$. **17.** $n\sec^n x\tan x$. **18.** $\dfrac{\sec^2 x}{2\sqrt{\tan x}}$.

19. $\dfrac{1}{2\sqrt{x}}\sec^2\sqrt{x}$. **20.** $\dfrac{1}{3}(\sin x)^{-\frac{2}{3}}\cos x$. **21.** $\dfrac{8x+3}{2\sqrt{4x^2+3x+1}}$.

22. $\dfrac{2ax+b}{2\sqrt{ax^2+bx+c}}$. **23.** $120(6x+1)^{19}$. **24.** $\dfrac{x}{\sqrt{a^2+x^2}}$.

25. $20(3x+1)(3x^2+2x+1)^9$. **26.** $-\dfrac{10x}{(x^2+1)^6}$. **27.** $na(ax+b)^{n-1}$.

28. $\dfrac{1}{3}a(ax+b)^{-\frac{2}{3}}$. **29.** $-\sin x.\cos(\cos x)$. **30.** $-\cos x.\sin(\sin x)$.

31. $\dfrac{1}{2}(3x+1)(3x^2+2x+1)^{-\frac{3}{4}}$. **32.** $x(1-x^2)^{-\frac{3}{2}}$. **33.** $\dfrac{\pi}{180}\cos x^\circ$.

34. $a\cos(ax+b)$. **35.** $\dfrac{a\cos(ax+b)}{2\sqrt{\sin(ax+b)}}$.

36. $12\sin^2(4x+5)\cos(4x+5)$. **37.** $na\sin^{n-1}(ax+b)\cos(ax+b)$.

38. $na\tan^{n-1}(ax+b)\sec^2(ax+b)$. **39.** $na\sec^n(ax+b)\tan(ax+b)$.

40. $\dfrac{(4x+5)\cos(2x^2+5x+6)}{2\sqrt{\sin(2x^2+5x+6)}}$. **41.** $3\sin x\cos x(a\sin x-b\cos x)$.

42. $15\sin x\cos x(a\sin x-b\cos x)(a\sin^3 x+b\cos^3 x)^4$.

43. $\dfrac{7\sin x\cos x}{\sqrt{3\sin^2 x-4\cos^2 x}}$. **44.** $-15\cos^4 3x\sin 3x$.

45. $6x\tan^2(x^2)\sec^2(x^2)$. **46.** $nbx^{n-1}\cos(a+bx^n)$.

47. $\dfrac{5\sec^2 2x+6\sec^2 2x\tan 2x}{\sqrt{5\tan 2x+3\sec^2 2x}}$. **48.** $mnx^{n-1}\sin^{m-1}(x^n)\cos(x^n)$.

49. $3\cos x-12\sin^2 x\cos x=3(4\cos^3 x-3\cos x)=3\cos 3x$.

50. $-12\cos^2 x\sin x+3\sin x=3(4\sin^3 x-3\sin x)=-3\sin 3x$.

51. $\cos u.\dfrac{du}{dx}$. **52.** $3\sin^2 u.\cos u\dfrac{du}{dx}$. **53.** $2u.\dfrac{du}{dx}$.

54. $nu^{n-1}\dfrac{du}{dx}$. **55.** $\dfrac{1}{2\sqrt{u}}\cdot\dfrac{du}{dx}$. **56.** $-\dfrac{1}{u^2}\cdot\dfrac{du}{dx}$.

57. $\dfrac{\cos u}{2\sqrt{\sin u}}\cdot\dfrac{du}{dx}$. **58.** $-n\cos^{n-1}u\,.\sin u\dfrac{du}{dx}$.

59. $\dfrac{1}{2\sqrt{u}}\cos\sqrt{u}\,.\dfrac{du}{dx}$. **60.** $\dfrac{u}{\sqrt{a^2+u^2}}\cdot\dfrac{du}{dx}$.

An arbitrary constant to be added from 61—75.

61. $\frac{1}{3}\sin^3 x$. **62.** $\dfrac{1}{n+1}\sin^{n+1}x$. **63.** $-\dfrac{1}{n+1}\cos^{n+1}x$.

64. $-\frac{1}{n}\cos nx$. **65.** $\frac{1}{n}\sin nx$. **66.** $\frac{1}{n}\tan nx$.

67. $\frac{1}{n}\tan nx - x$. **68.** $-\frac{1}{a}\cos(ax+b)$. **69.** $\frac{1}{a}\sin(ax+b)$.

70. $\dfrac{1}{(n+1)a}(ax+b)^{n+1}$. **71.** $\dfrac{1}{12}(x^2+4)^6$. **72.** $-\dfrac{180}{\pi}\cos x^\circ$.

73. $\frac{1}{5}\sec^5 x$. **74.** $-\dfrac{1}{x^2+3x+5}$. **75.** $2\sqrt{x^2+3x+5}$.

EXERCISES LXVII. p. 294.

1. $-\dfrac{2x}{(1+x^2)^{\frac{3}{2}}(1-x^2)^{\frac{1}{2}}}$. **2.** $\dfrac{a^2-2x^2}{\sqrt{a^2-x^2}}$. **3.** $-\dfrac{2\sin x}{(1+\sin^2 x)^{\frac{3}{2}}}$.

4. $2x\cos x(\cos x - x\sin x)$. **5.** $(2x+1)^6(3x+4)^4(72x+71)$.

6. $4(x^2+x+1)^3(2x^3-3x^2+4)(7x^4-4x^3-3x^2+5x+4)$.

7. $\dfrac{a^2}{(a^2-x^2)^{\frac{3}{2}}}$. **8.** $\dfrac{\sqrt{x}-1}{2\sqrt{x}\sqrt{1+x}(1+\sqrt{x})^2}$.

9. $(ax+b)^{m-1}(cx+d)^{n-1}\{(m+n)acx+mad+nbc\}$.

10. $3\sin^2 x\sin 4x$. **11.** $\dfrac{4(x^4-x^2-4)}{\sqrt{x^2-4}}$.

12. $\dfrac{-15\cos^2 5x\sin 5x - 12\sin^3 2x\cos 2x}{\sqrt{2\cos^3 5x - 3\sin^4 2x}}$. **13.** $n\sin^{n-1}x\sin(n+1)x$.

14. $n\sin^{n-1}x\cos(n+1)x.$ **15.** $u^{m-1}v^{n-1}\left(mv\dfrac{du}{dx}+nu\dfrac{dv}{dx}\right).$

16. $\sin^{m-1}u\cos^{n-1}v\left(m\cos u\cos v\dfrac{du}{dx}-n\sin u\sin v\dfrac{dv}{dx}\right).$

17. $\dfrac{5\sqrt{\sin 5x}\,(15\cos 5x+\cos 10x+2)}{2\,(5+\cos 5x)^{\frac{3}{2}}}.$ **18.** $\dfrac{n}{\sqrt{a^2+x^2}}(x+\sqrt{a^2+x^2})^n.$

19. $\dfrac{n}{2}\cdot\dfrac{\sin(n+1)x-(2n-1)\sin(n-1)x}{\sin^n x}.$ **20.** $\dfrac{1}{3}\left(\dfrac{u}{v}\right)^{-\frac{2}{3}}\left(\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}\right).$

EXERCISES LXVIII. p. 296.

1. $\dfrac{3+8t}{2}.$ **2.** $\dfrac{t^2}{1+2t^2}.$ **3.** $-\dfrac{a\cos\theta+b\sin\theta}{a\sin\theta-b\cos\theta}.$

4. $2m.$ **9.** $\dfrac{u\sin a}{g}$ seconds.

EXERCISES LXIX. p. 298.

1. $\dfrac{5}{3}x^2.$ **2.** $\sec^3 x.$ **3.** $\dfrac{2\cos 2x}{\cos x}.$ **4.** $\dfrac{4t-3}{6t+2}.$ **5.** $-\cot x.$

EXERCISES LXX. p. 299.

1. $\dfrac{\pi ab}{4}.$ **2.** $3\pi a^2.$ **3.** $\dfrac{\mathrm{M}b^2}{4};\ \dfrac{\mathrm{M}a^2}{4};\ \dfrac{\mathrm{M}(a^2+b^2)}{4}.$

4. $\pi a^3 w$ [w = weight of 1 c. ft. of water]; $\dfrac{a}{4}$ ft. below centre.

5. $\left(\dfrac{4a}{3\pi},\ \dfrac{4b}{3\pi}\right).$

EXERCISES LXXI. p. 300.

1. $x=\dfrac{4y+3}{2}.$ **2.** $\dfrac{4x+3}{2}.$

EXERCISES LXXIV. p. 309.

1. $-\frac{x}{y}$; $3x \pm 4y = 5a$. **2.** $-\frac{b^2x}{a^2y}$. **5.** $-\frac{ax+hy}{hx+by}$.

6. $2x - y - 5 = 0$. **7.** $-\frac{2\sin x}{3\sin y}$.

8. $-\frac{2x-3}{2y+10}$; $5x - 12y = 152$; $12x + 5y + 7 = 0$. **9.** $\frac{4x-3y-2}{3x-8y-1}$.

EXERCISES LXXVIII. p. 321.

3. (i) Ake^{kx}. (ii) $\frac{1}{k}e^{kx} + c$.

6. (i) $3e^{3x}$. (ii) $2x \cdot e^{x^2}$. (iii) $2e^{2x+3}$. (iv) ae^{ax+b}. (v) $\cos x \cdot e^{\sin x}$.
(vi) $e^x(x+1)$. (vii) $e^x(x^2+2x)$. (viii) $e^{kx}(kx^n + nx^{n-1})$.
(ix) $e^x(\sin x + \cos x)$. (x) $e^{2x}(2\sin 3x + 3\cos 3x)$.
(xi) $e^{ax}\{a\sin(bx+c) + b\cos(bx+c)\}$.
(xii) $e^{-\cdot 3x}\{-\cdot 3\sin 2x + 2\cos 2x\}$. (xiii) $(2x+1)e^{x^2+x}$.
(xiv) $e^u \cdot \frac{du}{dx}$. (xv) $e^x(x^3\sin x + 3x^2\sin x + x^3\cos x)$.
(xvi) $e^{3x}(3\sin^4 5x + 20\sin^3 5x\cos 5x)$. (xvii) $7^x\log_e 7 = 7^x \times 1\cdot 9459$.

7. [Add an arbitrary constant.]

(i) e^x. (ii) $\frac{1}{3}e^{3x}$. (iii) $\frac{1}{2}e^{2x+5}$. (iv) $\frac{1}{2}e^{x^2}$. (v) $e^{\tan x}$.

(vi) $-e^{\frac{1}{x}}$. (vii) $\frac{7^x}{\log_e 7} = 7^x \times \cdot 5139$.

8. $e^4 - e^2 = 47\cdot 209$; $2\cdot 36$ sq. ins. **10.** $-0\cdot 237$.

EXERCISES LXXIX. p. 325.

2. (i) $\frac{1}{x}$. (ii) $\frac{3}{x}$. (iii) $\frac{3}{3x+5}$. (iv) $-\tan x$. (v) $2\operatorname{cosec} 2x$.

(vi) $\frac{6x+5}{3x^2+5x+1}$. (vii) $2\cot 2x$. (viii) $\frac{\cdot 910}{x}\left[=\frac{1}{x\log_e 3}\right]$.

(ix) $\operatorname{cosec} x$. (x) $\sec x$. (xi) $\sec x$. (xii) $\dfrac{x}{x^2+1}$.

(xiii) $\dfrac{1}{x+1}+\dfrac{1}{x+2}+\dfrac{1}{x+3}$. (xiv) $\dfrac{1}{\sqrt{x^2+a^2}}$. (xv) $\dfrac{1}{\sqrt{x^2-a^2}}$.

(xvi) $\dfrac{2a}{a^2-x^2}$. (xvii) $-\dfrac{2a}{x^2-a^2}$. (xviii) $1+\log_e x$. (xix) $x+2x\log_e x$.

(xx) $\dfrac{a}{ax+b}$. (xxi) $\dfrac{2ax+b}{ax^2+bx+c}$. (xxii) $\dfrac{1}{u}\cdot\dfrac{du}{dx}$.

3. [An arbitrary constant should be added in each case.]

(i) $\log_e x$. (ii) $\dfrac{1}{2}\log_e x$ *or* $\log_e\sqrt{x}$. (iii) $\dfrac{1}{2}\log_e(2x+3)$.

(iv) $-\log_e\cos x$ *or* $\log_e\sec x$. (v) $\log_e\tan\dfrac{x}{2}$.

(vi) $\log_e\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$ *or* $\log_e(\sec x+\tan x)$. (vii) $\log_e(x^2+3x+5)$.

(viii) $\dfrac{1}{a}\log(ax+b)$. (ix) $\dfrac{1}{2}\log_e(x^2+4x-7)$. (x) $\log_e(x+\sqrt{x^2+a^2})$.

(xi) $\log_e(x+\sqrt{x^2-a^2})$. (xii) $\dfrac{1}{2a}\log_e\dfrac{a+x}{a-x}$. (xiii) $\dfrac{1}{2a}\log_e\dfrac{x-a}{x+a}$.

(xiv) $x\log_e x-x$. (xv) $\dfrac{x^2}{2}+2x+\log_e x$. (xvi) $\dfrac{x^2}{2}+3x-7\log_e(x+2)$.

(xvii) $\dfrac{x^3}{3}+x^2-x+5\log_e(x-3)$.

5. (i) and (ii) $\log_e 6=1{\cdot}792$. **6.** 219 ft. lbs. wt. **7.** 219 ft. lbs. wt.

8. (i) $\log_e 3=1{\cdot}0986$. (ii) $\dfrac{1}{3}\log_e\dfrac{11}{8}=0{\cdot}1062$. (iii) $\dfrac{1}{4a}\log_e 3=\dfrac{0{\cdot}2747}{a}$.

(iv) $\dfrac{1}{2}\log_e 2=0{\cdot}3466$. (v) $\log_e(\sqrt{2}+1)={\cdot}8813$.

9. $\left(\dfrac{3}{\log_e 4}, \dfrac{3}{8\log_e 4}\right)$ *or* $(2{\cdot}164, 0{\cdot}271)$.

10. $\left(\dfrac{4\log_e 4}{3}, \dfrac{7}{32}\right)$ *or* $(1{\cdot}8484, 0{\cdot}21875)$.

11. $\log_e 2=0{\cdot}6931$. **12.** 3030.

13. Max. value $\dfrac{1}{2}(-11+30\log_e 3)=10{\cdot}98$, when $x=1$.

Min. value $\dfrac{1}{2}(-27+30\log_e 5)=10{\cdot}64$, when $x=3$.

EXERCISES LXXXI. p. 334.

1. $\dfrac{(3x+1)^6(4x-3)^4}{(5x+2)^6}(480x^2+173x-26)$.

2. $\dfrac{x^3+1}{2(x+2)^{\frac{1}{2}}(3x+1)^{\frac{4}{3}}}(37x^4+81x^3+24x^2+x-3)$.

3. $\dfrac{(2x-3)^{\frac{1}{2}}}{2(3x^2+1)^{\frac{1}{2}}(4x^2+2x+1)^{\frac{3}{2}}}(96x^5+24x^4-19x^2+32x+12)$.

4. $\dfrac{e^{\sin x}.\cos^3 x}{(4x+1)^2}\{(4x+1)(\cos x-3\tan x)-4\}$.

5. $x^x(1+\log_e x)$. 6. $\dfrac{x^{\frac{1}{x}}(1-\log_e x)}{x^2}$.

7. $(\sin x)^{\cos x}\left(\dfrac{\cos^2 x}{\sin x}-\sin x.\log_e \sin x\right)$.

EXERCISES LXXXII. p. 340.

1. $x=2e^{3t}$; 16206 ft.; 1·304 secs. 2. 81·1 lbs. wt.; $\angle \mathrm{AOP}=56° 44'$.

3. After 6·575 minutes.

4. $k=-\dfrac{cg}{l}$; (i) 2 mins., (ii) 3 mins., (iii) 3·32 mins.

5. $v-850=(u-850)e^{-kt}$. [u ft./sec. is speed when $t=0$.]

6. 48·634 ft. [$s=50(1-e^{-\cdot 03t})$.]

7. $y=\dfrac{1}{4}e^{\frac{3x}{12800}}$. [$y$ sq. ins. is cross-section x ins. from the lower end.]

EXERCISES LXXXIII. p. 342.

3. (i) $y=\mathrm{A}\cos 2x+\mathrm{B}\sin 2x$.
(ii) $y=\mathrm{A}e^{2x}+\mathrm{B}e^{-2x}$ *or* $\mathrm{A}'\cosh 2x+\mathrm{B}'\sinh 2x$.

4. $y=5\cos 2x+2\sin 2x$. 5. $y=\dfrac{7}{2}e^{2x}+\dfrac{3}{2}e^{-2x}$.

6. (i) 60 ft./sec. (ii) 75 ft./sec.

10. (i) $A \cos \sqrt{q-p^2} \,.\, x + B \sin \sqrt{q-p^2} \,.\, x$. (ii) $Ae^{\sqrt{p^2-q}\,.\,x} + Be^{-\sqrt{p^2-q}\,.\,x}$.
(iii) $Ax + B$.

11. (i) $y = Ae^{-x} + Be^{-3x}$. (ii) $y = e^{-2x}(A \cos x + B \sin x)$.
(iii) $y = e^{-2x}(Ax + B)$. (iv) $y = Ae^{4x} + Be^{x}$.
(v) $y = e^{\frac{1}{2}x}\left(A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x\right)$. (vi) $y = e^{\frac{1}{2}x}(Ax + B)$.

12. -3. **13.** 8. **14.** $-\frac{7}{68}, -\frac{6}{68}$. **16.** $\frac{3}{4}$.

18. (i) $y = Ae^{-x} + Be^{-3x} + 2$. (ii) $y = e^{\frac{1}{2}x}\left(A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x\right) + \frac{11}{7}$.
(iii) $y = e^{\frac{1}{2}x}(Ax + B) + \frac{4}{5}$.

MISCELLANEOUS EXAMPLES ON CHAPTERS IX—XI.

p. 344.

A.

2. $\frac{9}{\sqrt{2}} - 2 + \frac{\pi}{4} = 5{\cdot}149$; $\frac{3}{2} + \log_e 2 = 2{\cdot}193$. **4.** 2M ft. lbs.

5. ·7033; ·7034.

B

1. (i) $6 \sin 3x \cos 3x$. (ii) $-\frac{1}{2} x^{-\frac{3}{2}}(8x \sin 4x + \cos 4x)$. (iii) $\frac{1}{2} \tan \frac{x}{2}$.
(iv) $e^{-3x}\left[-3 \sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)\right]$. (v) $\frac{x-y}{x+3y}$.

2. (i) $-\frac{1}{3} e^{-3x} + c$. (ii) $-\frac{2}{3} \cos 3x + \frac{2}{3} \sin 6x + c$. (iii) 1.

3. 4 ft./sec.; $4\frac{1}{8}$ ft./sec.^{2}; both towards O if A is moving away from O.

4. $\frac{1}{me}$ when $x = \frac{1}{m}$.

C.

1. $-\frac{\sqrt{a^2 - x^2}}{x}$; a. **2.** $\frac{1}{2}$; ·500. **3.** ·70″; ·87 ft.

D.

1. (i) $\dfrac{2-2x^2}{x^4+x^2+1}$. (ii) $\dfrac{1}{2x(1+x)^{\frac{1}{2}}(1-x)^{\frac{3}{2}}}$.

2. $(2, 1), (-1, 1)$; $2x-3y-1=0$, $x-3y+4=0$. 3. $\sqrt{13}-2=1{\cdot}606$.

4. $\cdot 8933r$. 5. $\dfrac{\pi b}{a}(1-e^{-aR^2})$; $14{\cdot}2$.

E.

1. (i) $\log_e x+1$, $\dfrac{1}{x}$. (ii) $e^x(\sin x+\cos x)$, $2e^x\cos x$.

(iii) $\dfrac{x\cos x-\sin x}{x^2}$, $\dfrac{-x^2\sin x-2x\cos x+2\sin x}{x^3}$.

(iv) $4\sin^3 x\cos x$, $4\sin^2 x(3\cos^2 x-\sin^2 x)$.

2. $-b\tan C.\omega$ (where b is CA and C the angle ACB).

3. $-\tan\theta$, $\dfrac{1}{3a\cos^4\theta\sin\theta}$.

4. $\dfrac{a^{\frac{3}{2}}}{a^{\frac{3}{2}}+b^{\frac{3}{2}}}c$ from centre of first sphere, $\dfrac{b^{\frac{3}{2}}}{a^{\frac{3}{2}}+b^{\frac{3}{2}}}c$ from centre of second.

5. $2\left(\dfrac{80}{\pi}+2\pi\right)=63{\cdot}496$.

F.

1. $\dfrac{5x^2-6x-2}{(x^2-x+1)^2}$, $-6\cos^2 2x\sin 2x$, $e^{ax}(a\sin bx+b\cos bx)$, $4\sec 4x$.

2. c; $PG=\dfrac{PN^2}{c}$. 3. $25{\cdot}1$ ft./sec., $24{\cdot}1$ ft./sec., $18{\cdot}85$ ft./sec.

5. $\pi G\cos a$ ft. lbs.

G.

1. (i) $\dfrac{2+2x-x^2}{(1-x)^2}$. (ii) $-3x(2-3x^2)^{-\frac{1}{2}}$. (iii) $\dfrac{2-2x}{2x-x^2}$.

(iv) $4\sec^2 4x$. (v) $e^{-3x}(1-3x)$. (vi) $\dfrac{1}{\sqrt{4a^2-x^2}}$.

2. $\dfrac{a}{\sqrt{2}}$. 4. (i) $\dfrac{38}{3}$. (ii) $0{\cdot}3054$. (iii) $0{\cdot}2456$. (iv) $\dfrac{1}{3}$.

(v) $0{\cdot}0429$. (vi) $0{\cdot}3466$.

5. $\dfrac{6M}{g}\left(\dfrac{d\theta}{dt}\right)^2$ ft. lbs.

H.

1. (i) $5x^4-8x^3+21x^2+2x-2$. (ii) $\dfrac{x^4-4x^3+21x^2-2x+2}{(x^2-2x+7)^2}$.

(iii) $144x^5-280x^4+168x^3-120x^2+78x-15$. (iv) $5\sec^2 5x$.

(v) $\dfrac{1}{(1-x^2)^{\frac{3}{2}}}$. (vi) $6\sec^2 3x\tan 3x$. (vii) $\dfrac{a^4+a^2x^2-4x^4}{\sqrt{a^2-x^2}}$.

2. $-0{\cdot}3690$ and $-1{\cdot}7602$ Minimum.
$0{\cdot}3690$ and $1{\cdot}7602$ Maximum.

3. 38·2 ins./sec. **4.** $6e^{-\frac{1}{2}t}\left(3\cos 3t-\frac{1}{2}\sin 3t\right)$. **5.** $\dfrac{b}{2}$.

I.

2. $\dfrac{d\phi}{dt}=\dfrac{\pi n}{60}\cdot\dfrac{\cos\theta}{\cos\phi}$; $\dfrac{dx}{dt}=-\dfrac{a\pi n}{30}\cdot\dfrac{\sin(\theta+\phi)}{\cos\phi}$; 1·714 radians/sec.; $-3{\cdot}715$ ft./sec.

4. 6·28 ft. lbs. wt., 6·98 ft. lbs. wt.

J.

2. 1·8083. **3.** $c^2\left(1+\dfrac{1}{e^2}\right)=c^2\times 1{\cdot}1353$.

4. $\dfrac{2r\sin^3 a}{3(a-\sin a\cos a)}$; $\dfrac{2r\sin a}{3a}$; $\dfrac{r\sin a}{a}$ from centre.

5. $\sqrt{\dfrac{216}{125}}=1{\cdot}3145$ ft.

K.

2. $x+25y+16=0$. $25x-y-100\frac{4}{5}=0$.

5. $2a^2$; $(1{\cdot}513a,\ 1{\cdot}003a)$.

L.

2. $2,\ 2,\ -\dfrac{5}{3}$. **3.** $a^2(\sqrt{2}-1)=0{\cdot}4142a^2$.

4. $\left(-6,\ \dfrac{\sqrt{3}}{2}\right)$; $\dfrac{\pi}{3}$ ft./sec. making 210° with OX.

$\dfrac{\pi^2\sqrt{52}}{18}$ (3·96) ft./sec.^{2} making $-13°\ 54'$ with OX.

5. 59·3 ft. lbs. wt.

M.

1. $6\tan 3x \sec^2 3x$, $x(1+2\log 4x)$, $-\dfrac{5a}{2(a-2x)^{\frac{1}{2}}(a+3x)^{\frac{3}{2}}}$,

$2(5x+3)^9(2x^2+7)^{11}(170x^2+72x+175)$, $3x(a^2+x^2)^{\frac{1}{2}}$, $\dfrac{3a}{a^2+9x^2}$.

2. $62° 45'$; $\dfrac{1}{\pi}=0{\cdot}3183$.

3. $11{\cdot}15$ ft. $\left(\dfrac{20\sqrt[3]{2}}{\sqrt[3]{2}+1}\right)$ from A. $8{\cdot}85$ ft. $\left(\dfrac{20}{\sqrt[3]{2}+1}\right)$ from B.

4. $A=100\sqrt{13}=360{\cdot}6$. $q=\tan^{-1}1{\cdot}5=$ C.M. of $56° 19'$.

5. $-a\omega\sin\omega t$, $-a\omega^2\cos\omega t$.

N.

1. $\dfrac{hx}{a^2}+\dfrac{ky}{b^2}=1$; $\dfrac{kx}{b^2}-\dfrac{hy}{a^2}=hk\left(\dfrac{1}{b^2}-\dfrac{1}{a^2}\right)$; $\dfrac{a^2}{h}$; $\dfrac{a^2-b^2}{a^2}h$; a^2-b^2.

2. $\dfrac{1}{20}$ radians/sec. **4.** (i) $-{\cdot}350$. (ii) $\dfrac{1}{3}$. (iii) $\dfrac{\pi}{16}$.

(iv) ${\cdot}0055$. (v) ${\cdot}0055$.

6. $3{\cdot}165a^3$.

O.

1. (i) $-\dfrac{6x^2+6xy+5y+7}{3x^2+3y^2+5x+8}$. (ii) $-\dfrac{2x}{\cos y+2y}$.

(iii) $(12x^3+18x+5)(6x^2+3)$. (iv) $y\cos x$ or $e^{\sin x}.\cos x$.

(v) $\dfrac{6y-7}{3}$. (vi) $\dfrac{1}{\cos x}$. (vii) $-\cot x$.

4. $20\log_e 3=21{\cdot}97$; $(7{\cdot}28, 1{\cdot}52)$; $1{\cdot}84$.

5. (i) $\dfrac{2}{\pi}$. (ii) $\dfrac{2}{\pi}$. (iii) $\dfrac{1}{\pi}$. (iv) $\dfrac{1}{2}$. (v) $\dfrac{1}{2}$.

P.

1. (i) $x^2\cos x+2x\sin x$. (ii) $2xe^{2x}(x+1)$.

(iii) $e^{ax}(a\cos^2 bx-2b\cos bx\sin bx)$. (iv) $\dfrac{x\sec^2 x-\tan x}{x^2}$.

(v) $\dfrac{\sin x+x\cos x}{2\sqrt{x\sin x}}$. (vi) $-\dfrac{1}{2}\sin\dfrac{x}{2}$. (vii) $\dfrac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$.

2. $3x+y=6$. **5.** 16000 ft. lbs. wt.

Q.

1. $\pm\sqrt{3x+2y}=5.$ **5.** $\left[\frac{3}{5}(b-a),\ 0\right].$

R.

1. $-1{\cdot}482$ ft./sec.; $6{\cdot}187$ ft./sec.^{2}

4. (i) $\frac{3}{5}$. (ii) $\cdot 0243$. (iii) 0. (iv) 0.

S.

2. $1\frac{1}{3}$ ins./sec.; $\frac{1}{36}$ radians/sec. **4.** $e^{\frac{1}{2}}=1{\cdot}649.$

5. $\frac{r^2}{4c}$ below centre.

T.

1. (i) $\frac{x^3}{6}+\frac{a^3}{2}\log_e x+c.$ (ii) $\log_e \sin x+c$ *or* $\log_e \mathrm{A}\sin x.$

(iii) $\frac{\pi}{4}$. (iv) $1-\frac{\pi}{4}=0{\cdot}2146.$ (v) $\frac{2}{27}(3x^3+1)^{\frac{3}{2}}+c.$

2. 8 ins./min.; $3{\cdot}175$ ins./min.

3. $\frac{3am(2-m^3)}{(1+m^3)^2}$; $\frac{3a(1-2m^3)}{(1+m^3)^2}$; $\frac{m(2-m^3)}{1-2m^3}$; $y=\frac{5}{4}x-a$; $y=-x+3a.$

4. $6\frac{mn^2(\mathrm{A}^2+\mathrm{B}^2)}{12g}$, $4\frac{mn^2(\mathrm{A}^2+\mathrm{B}^2)}{12g}$, $3\frac{mn^2(\mathrm{A}^2+\mathrm{B}^2)}{12g}$. **5.** $7{\cdot}82$ ins.

EXERCISES LXXXIV. p. 373.

6. $2{\cdot}095$. **7.** $-0{\cdot}315$, $0{\cdot}446$, $1{\cdot}069$. **8.** $0{\cdot}4142$. **9.** $0{\cdot}85138$.

10. $12{\cdot}03$. **11.** $11{\cdot}34$ ins. **12.** $1{\cdot}1839$. **13.** $1{\cdot}80$, $-1{\cdot}13$.

14. $4{\cdot}167$ ins. **15.** 28° 23′. **16.** $1{\cdot}166$. **17.** $2{\cdot}289$.

18. $0{\cdot}657$. **19.** $4{\cdot}536$. **20.** $3{\cdot}945$.

EXERCISES LXXXV. p. 379.

[Add an arbitrary constant in each case.]

1. $-\frac{1}{6}\cos^6 x$. **2.** $-\frac{1}{2}\cos x^2$. **3.** $\log_e(x^2+3x+5)$.

4. $\frac{2}{\sqrt{11}}\tan^{-1}\frac{2x+3}{\sqrt{11}}$. **5.** $\log_e(x^2+3x+5)+\frac{2}{\sqrt{11}}\tan^{-1}\frac{2x+3}{\sqrt{11}}$.

6. $2\log_e(x^2+3x+5)+\frac{10}{\sqrt{11}}\tan^{-1}\frac{2x+3}{\sqrt{11}}$. **7.** $-\cos(\log_e x)$.

8. $e^{\sin^{-1}x}$. **9.** $\log_e(1+\log_e x)$.

10. $\frac{1}{6}\tanh^{-1}\frac{2x}{3}$ *or* $\frac{1}{12}\log_e\frac{3+2x}{3-2x}$.

11. $-\frac{1}{6}\coth^{-1}\frac{2x}{3}$ *or* $-\frac{1}{12}\log_e\frac{2x+3}{2x-3}$. **12.** $\frac{1}{6}\tan^{-1}\frac{2x}{3}$.

13 $\sin x-\frac{1}{3}\sin^3 x$. **14.** $\frac{1}{6}\tan^6 x$. **15.** $\frac{1}{2}\tan^2 x+\log_e\cos x$.

16. $\frac{2}{3}(2a+x)^{\frac{3}{2}}$. **17.** $-\frac{\cos^3 x}{3}+\frac{\cos^5 x}{5}$.

18. (i) $-2\coth^{-1}(2x-3)$ *or* $\log_e\frac{x-2}{x-1}$.

(ii) $-2\tanh^{-1}(2x-3)$ *or* $\log_e\frac{2-x}{1-x}$.

19. $\frac{3}{2}\log_e(x^2+1)+\tan^{-1}x$. **20.** $-\frac{1}{a}\log_e\cos(ax+b)$.

21. $-\frac{2}{3}\cos^3 x+\cos x$. **22.** $-\frac{1}{3}\sqrt{2-3x^2}$.

23. $\frac{2}{\sqrt{3}}\tanh^{-1}\frac{x}{\sqrt{3}}+\frac{1}{2}\log_e(3-x^2)$. **24.** $\frac{1}{3}\log_e(x^3+1)$.

25. $\frac{1}{4\cos^4 x}$. **26.** $-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}$. **27.** $(\log_e x)^2$.

28. $2\tanh^{-1}\sqrt{x}$. **29.** $\frac{1}{2}(\tan^{-1}x)^2$.

30. $\log_e 2\,.\log_e x+\frac{1}{2}(\log_e x)^2$.

EXERCISES LXXXVI. p. 381.

1. $\frac{1}{2a^3}\left(\tan^{-1}\frac{x}{a}+\frac{ax}{a^2+x^2}\right)$. **2.** $-\frac{3x^2-2x+6}{6(x-3)^4}$.

3. $\frac{1}{4a^3}\left[\log\frac{x+a}{x-a}-\frac{2ax}{x^2-a^2}\right]$ *or* $\frac{1}{2a^3}\left[\coth^{-1}\frac{x}{a}-\frac{ax}{x^2-a^2}\right]$.

4. $\frac{9}{4}\left(\sin^{-1}\frac{2x}{3}+\frac{2x}{9}\sqrt{9-4x^2}\right)$. **5.** $\frac{a^4}{8}\sin^{-1}\frac{x}{a}-\frac{x}{8}(a^2-2x^2)\sqrt{a^2-x^2}$.

6. $-\frac{(a^2-x^2)^{\frac{3}{2}}(2a^2+3x^2)}{15}$. **7.** $\frac{x^2+10x-5}{x+5}-10\log_e(x+5)$.

8. $\frac{a^2}{2}\sinh^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2+x^2}$. **9.** $\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}$.

10. $-\frac{1}{a}\log_e\frac{a+\sqrt{a^2-x^2}}{x}$. **11.** $\frac{x^3+3x^2+12x-44}{3}+8\log_e(x-2)$.

12. $\frac{1}{16}\left[\frac{16x^3+36x^2+108x-397}{6}+27\log_e(2x-3)\right]$. **13.** $0{\cdot}212a^3$.

14. $\sinh^{-1}\frac{x}{a}$. **15.** $\frac{1}{2}[\log_e(\sec x+\tan x)+\tan x\sec x]$.

EXERCISES LXXXVII. p. 383.

1. $\frac{x^2}{2}\log_e x-\frac{x^2}{4}$. **2.** $e^x(x-1)$. **3.** $e^x(x^2-2x+2)$.

4. $x\tan^{-1}x-\frac{1}{2}\log_e(1+x^2)$. **5.** $x\sin^{-1}x+\sqrt{1-x^2}$. **6.** $-x\cos x+\sin x$.

7.
$$-\frac{1}{2}\left[\frac{1}{m+n}\left(x\cos(m+n)x-\frac{1}{m+n}\sin(m+n)x\right)\right.$$
$$\left.+\frac{1}{m-n}\left(x\cos(m-n)x-\frac{1}{m-n}\sin(m-n)x\right)\right].$$

8. $\frac{x^4}{4}\log_e x-\frac{x^4}{16}$. **9.** $-\frac{1}{2x^2}\log_e x-\frac{1}{4x^2}$. **10.** $x\tan x+\log_e\cos x$.

12. $\frac{1}{a^2+b^2}\cdot e^{ax}[a\sin(bx+c)-b\cos(bx+c)]$;

$\frac{1}{a^2+b^2}\cdot e^{ax}[b\sin(bx+c)+a\cos(bx+c)]$.

14. $\frac{16}{35}$; $\frac{35\pi}{256}$. $\left[\frac{6.4.2}{7.5.3};\ \frac{7.5.3.1}{8.6.4.2}\cdot\frac{\pi}{2}\right]$.

MISCELLANEOUS EXAMPLES ON CHAPTER XIII. p. 385.

1. $\frac{5}{2}\log_e(x^2+4)+\frac{1}{2}\tan^{-1}\frac{x}{2}$. **2.** $\frac{x^2+1}{2}\tan^{-1}x-\frac{x}{2}$.

3. $\frac{x^3}{3}-\frac{x^2}{2}+2x-2\log_e(x+1)$. **4.** $\frac{1}{3}\tan^{-1}x^3$.

5. $\log_e a\,.\log_e x+\frac{1}{2}(\log_e x)^2$. **6.** $\log_e(1+\sin x)$.

7. $-\frac{\sqrt{a^2-x^2}}{a^2x}$. **8.** $\frac{1}{3}\sqrt{a^2+x^2}\,.(x^2-2a^2)$.

9. $\frac{1}{\sqrt{2}}\sinh^{-1}\frac{4x+3}{\sqrt{31}}$ *or* $\frac{1}{\sqrt{2}}\log_e\frac{4x+3+2\sqrt{2(2x^2+3x+5)}}{\sqrt{31}}$.

10. $\frac{1}{5}x^{\frac{5}{2}}-5x^{\frac{1}{2}}$. **11.** $\frac{1}{15}\sqrt{x-5}\,(3x^2+20x+125)$.

12. $\frac{1}{8}\sin^4 2x$. **13.** $-\frac{1}{3}x\cos 3x+\frac{1}{9}\sin 3x$.

14. $-\frac{1}{3}x^2\cos 3x+\frac{2}{9}x\sin 3x+\frac{2}{27}\cos 3x$. **15.** $-\frac{3}{10}(3-2x)^{\frac{5}{3}}$.

16. $-\frac{1}{160}(3-2x)^{\frac{5}{3}}(27+30x)$. **17.** $\frac{1}{9}\log_e(2-3x)+\frac{2}{9(2-3x)}$.

18. $\frac{4}{\pi}\log_e\sec\left(\frac{\pi x}{4}+a\right)$. **19.** $\frac{2}{\sqrt{3}}\tanh^{-1}\frac{x}{\sqrt{3}}+\frac{1}{2}\log_e(3-x^2)$.
or $\left(\frac{1}{2}+\frac{1}{\sqrt{3}}\right)\log_e(\sqrt{3}+x)+\left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)\log_e(\sqrt{3}-x)$.

20. $\frac{4x^3-24x^2+3x+45}{(x-2)^2}+24\log_e(x-2)$. **21.** $\log_e t=\log_e\tan\frac{x}{2}$.

22. $-\log_e\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)=\log_e\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$. **23.** $\theta=\sin^{-1}\frac{x}{a}$.

24. $\frac{\theta}{a}=\frac{1}{a}\tan^{-1}\frac{x}{a}$. **25.** 1·568. **26.** 0·430. **27.** −0·148.

28. $\frac{16}{35}$. **29.** 0·0266. **30.** 0·430 $\left[\frac{35\pi}{256}\right]$.

31. $\frac{2}{3}\tan^{-1}\frac{1}{3}=0{\cdot}2145$. **32.** $\frac{1}{3}\log_e 2=0{\cdot}2310$. **33.** $y=\frac{w}{2T_0}\,.\,x^2$.

35. $\dfrac{m}{abg}\tan^{-1}\dfrac{abu}{2a^2+b^2u^2}$. **36.** $v=v_1\tanh\dfrac{gt}{v_1}$. **37.** $\dfrac{\pi Ra}{4}$ ft. lbs. wt.

38. If v ft./sec. is speed, and t secs. time,

$$a=\frac{110}{7},\quad b=\frac{9}{3080},\quad A=\frac{198}{35}.$$

41·77 m./hr.; 49·9998 m./hr.

EXERCISES LXXXIX. p. 393.

2. (i) $\tan\theta$. (ii) $-\cot\theta$. (iii) $\cot\theta$. (iv) $-\tan\theta$.

4. $\dfrac{\pi a^2}{4}$. **5.** $\dfrac{\pi a^2}{8}$. **6.** $\dfrac{a^2}{2}$. **8.** $\cot\phi=k$.

10. $r=ae^{k\theta}$ is the same as $r=e^{k\left(\theta+\frac{\log_e a}{k}\right)}$.

$r=be^{k\theta}$ is the same as $r=e^{k\left(\theta+\frac{\log_e b}{k}\right)}$.

Table of the Exponential and Hyperbolic Functions of Numbers from 0 to 2·5, at Intervals of ·1.

x	e^x	e^{-x}	$\cosh x$	$\sinh x$	$\tanh x$
0	1·000	1·000	1·000	0	0
·1	1·105	·905	1·005	·100	·100
·2	1·221	·819	1·020	·201	·197
·3	1·350	·741	1·045	·305	·291
·4	1·492	·670	1·081	·411	·380
·5	1·649	·607	1·128	·521	·462
·6	1·822	·549	1·185	·637	·537
·7	2·014	·497	1·255	·759	·604
·8	2·226	·449	1·337	·888	·664
·9	2·460	·407	1·433	1·027	·716
1·0	2·718	·368	1·543	1·175	·762
1·1	3·004	·333	1·669	1·336	·801
1·2	3·320	·301	1·811	1·509	·834
1·3	3·669	·273	1·971	1·698	·862
1·4	4·055	·247	2·151	1·904	·885
1·5	4·482	·223	2·352	2·129	·905
1·6	4·953	·202	2·577	2·376	·922
1·7	5·474	·183	2·828	2·646	·935
1·8	6·050	·165	3·107	2·942	·947
1·9	6·686	·150	3·418	3·268	·956
2·0	7·389	·135	3·762	3·627	·964
2·1	8·166	·122	4·144	4·022	·970
2·2	9·025	·111	4·568	4·457	·976
2·3	9·974	·100	5·037	4·937	·980
2·4	11·023	·091	5·557	5·466	·984
2·5	12·182	·082	6·132	6·050	·987

Table of Logarithms to Base *e*.

	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9
1	0	·095	·182	·262	·336	·405	·470	·531	·588	·642
2	·693	·742	·788	·833	·875	·916	·956	·993	1·030	1·065
3	1·099	1·131	1·163	1·194	1·224	1·253	1·281	1·308	1·335	1·361
4	1·386	1·411	1·435	1·459	1·482	1·504	1·526	1·548	1·569	1·589
5	1·609	1·629	1·649	1·668	1·686	1·705	1·723	1·740	1·758	1·775
6	1·792	1·808	1·825	1·841	1·856	1·872	1·887	1·902	1·917	1·932
7	1·946	1·960	1·974	1·988	2·001	2·015	2·028	2·041	2·054	2·067
8	2·079	2·092	2·104	2·116	2·128	2·140	2·152	2·163	2·175	2·186
9	2·197	2·208	2·219	2·230	2·241	2·251	2·262	2·272	2·282	2·293

$\log 10 = 2{\cdot}303$, $\log 10^2 = 4{\cdot}605$, $\log 10^3 = 6{\cdot}908$.

INDEX

The numbers refer to pages.

CAMBRIDGE: PRINTED BY JOHN CLAY, M.A. AT THE UNIVERSITY PRESS.

A Selection of the Books on Mathematics, Physics and Chemistry

published by the

Cambridge University Press

A complete catalogue of the publications of the Cambridge University Press will be sent on application

ALGEBRA

Elementary Algebra. By W. W. Rouse Ball, M.A. Extra fcap. 8vo. 4s 6d

Algebra for Secondary Schools. By Charles Davison, Sc.D. Crown 8vo. 6s. Or in two volumes, with Answers, 3s each; without Answers, 2s 6d each

Exercises separately. With Answers, 3s. Without Answers, 2s 6d

Problem Papers supplementary to *Algebra for Secondary Schools.* By C. Davison, Sc.D. 8d

Higher Algebra for Colleges and Secondary Schools. By C. Davison, Sc.D. Demy 8vo. 6s

An Algebra for Preparatory Schools. By Trevor Dennis, M.A. Crown 8vo. 2s. Teachers' Edition, with Answers interleaved. 5s net

Elementary Algebra. By C. H. French, M.A., and G. Osborn, M.A. Crown 8vo. With answers, 4s 6d. Without answers, 4s

Elementary Algebra. By C. Godfrey, M.V.O., M.A., and A. W. Siddons, M.A.

Prices :—	*With Answers*	*Without Answers*
Complete in one volume	4s 6d	4s
Or in two volumes :—		
Volume I, Algebra for Beginners ...	2s 6d	2s
Volume II	2s 6d	2s
Volume I is also published in two parts :—		
Part I, Chapters I—IV	1s	1s
Part II, Chapters V—XIV ...	2s	1s 6d

A School Algebra Course. By F. Gorse, M.A. With or without Answers. Crown 8vo. 3s. Or in three parts. 1s each

ARITHMETIC

The Winchester Arithmetic. By C. Godfrey, M.V.O., M.A., and G. M. Bell, B.A. Crown 8vo. 3s. Or in 2 parts, 1s 6d each. Teachers' Edition, with Answers, 6s net

Arithmetic for Schools. By C. Smith, M.A. Extra fcap. 8vo. With or without Answers. Stereotyped Edition. 3s 6d. Or in two parts, 2s each

SOLUTIONS OF THE EXAMPLES. By G. Hale, M.A. 7s 6d

GEOMETRY

A Shorter Geometry. By C. GODFREY, M.V.O., M.A., and A. W. SIDDONS, M.A. Crown 8vo. 2s 6d. Or in Four Parts:

Part I. (First and Second Stages, Geometry for Beginners), 1s.
Part II. (Third Stage, Books I. and II.). III. (Third Stage, Book III. The Circle). IV. (Third Stage, Book IV. Similarity), 10d each

NOTES AND ANSWERS TO EXERCISES. 6d

Solutions of the Exercises. By E. A. PRICE, B.A. Crown 8vo. 4s 6d net

Elementary Geometry (Practical and Theoretical). By the same authors. Crown 8vo.

Complete in one volume. 3s 6d

Or in two parts: Part I, Experimental Geometry, 1s; Part II. Theoretical Geometry, 3s. Part II is also published in four separate books:—Book I; Book II (Area); Book III (The Circle); Book IV (Similarity); 1s each

Or in two volumes, 2s each. Vol. I contains Part I (Experimental Geometry) and Part II, Books I and II; Vol. II contains Part II, Books III and IV.

ANSWERS TO THE EXAMPLES. 4d post free

SOLUTIONS OF THE EXERCISES. By E. A. PRICE, B.A. Crown 8vo. 5s net

Solid Geometry. By the same authors. Crown 8vo. 1s 6d

SOLUTIONS OF THE EXERCISES. By C. L. BEAVAN, M.A. Crown 8vo. 5s net

Elementary Geometry together with Solid Geometry. By C. GODFREY and A. W. SIDDONS. Crown 8vo. 4s 6d

A Shorter Geometry together with Solid Geometry. By the same authors. Crown 8vo. 4s

Modern Geometry. By the same authors. Crown 8vo. 4s 6d

SOLUTIONS OF THE EXERCISES. 4s net

Plane Geometry for Secondary Schools. By CHARLES DAVISON, Sc.D., and C. H. RICHARDS, M.A. Crown 8vo. 4s. Or in four parts, paper boards. 1s each. The four parts are divided as follows:—Book I.—Triangles and Parallelograms. Book II.—Areas. Book III.—The Circle. Book IV.—Proportion applied to Geometrical Magnitudes.

The Elements of Solid Geometry. By CHARLES DAVISON, Sc.D. Crown 8vo. 2s 6d net

A Geometry for Schools. By F. W. SANDERSON, M.A., and G. W. BREWSTER, M.A. Published in two forms, with or without Answers. Crown 8vo. 2s 6d

Examples only, with or without Answers. 1s 6d

Geometrical Drawing. With Notes and Examples. By W. H. BLYTHE, M.A. Fcap. 8vo. Part I. Plane and Elementary Solid. 1s 6d. Part II. Solid or Descriptive Geometry. 1s 6d

The Contents of the Fifth and Sixth Books of Euclid. Arranged and explained by M. J. M. HILL, M.A., D.Sc., F.R.S. Second edition. Crown 4to. 6s net

Euclid. Elements of Geometry. Edited by H. M. TAYLOR, M.A.

Books I, II. 1s 6d	Books I—IV. 3s
Books III, IV. 1s 6d	Books I—VI. 4s
Books V, VI. 1s 6d	Books I—VI and XI, XII. In
Books XI, XII. 1s 6d	One Vol. complete. 5s

SOLUTIONS OF THE EXERCISES. By W. W. TAYLOR, M.A.

Books I—IV. 6s. Books VI and XI. 6s. Books I—IV, VI and XI. 10s 6d

Analytical Geometry. A First Course. By C. O. TUCKEY and W. A. NAYLER. Crown 8vo. 5s net

An Elementary Treatise on Cross-Ratio Geometry. With Historical Notes. By the Rev. JOHN J. MILNE, M.A. With 129 text-figures. Crown 8vo. 6s

Elementary Projective Geometry. By A. G. PICKFORD, M.A., M.Sc. Crown 8vo. 4s

A Course of Pure Geometry. By E. H. ASKWITH, D.D. Crown 8vo. 5s

TRIGONOMETRY

A Class Book of Trigonometry. By C. DAVISON, Sc.D. Crown 8vo. 3s

An Elementary Treatise on Plane Trigonometry. By E. W. HOBSON, Sc.D., F.R.S., and C. M. JESSOP, M.A. Extra fcap. 8vo. 4s 6d

Plane Trigonometry. By S. L. LONEY, M.A. Ninth edition. Crown 8vo. 7s 6d; or in separate Parts—Part I, an Elementary Course, excluding the use of imaginary quantities, 5s; Part II, Analytical Trigonometry, 3s 6d

SOLUTIONS OF THE EXAMPLES. Parts I and II. 6s each

The Elements of Trigonometry. By S. L. LONEY, M.A. Third edition. Extra fcap. 8vo. 3s 6d

Trigonometry for Beginners. By J. W. MERCER, M.A. Crown 8vo. 4s

Treatise on Elementary Dynamics. By S. L. Loney, M.A. Seventh edition. Crown 8vo. 7s 6d
SOLUTIONS. 7s 6d

The Elements of Hydrostatics. By S. L. Loney, M.A. Extra fcap. 8vo. Fourth edition. 4s 6d
SOLUTIONS. 5s

An Elementary Treatise on the Dynamics of a Particle and of Rigid Bodies. By S. L. Loney, M.A. Demy 8vo. 12s

An Elementary Treatise on Statics. By S. L. Loney, M.A. Demy 8vo. 12s

Elements of the Differential and Integral Calculus. By A. E. H. Love, M.A., D.Sc., F.R.S. Crown 8vo. 5s

Theoretical Mechanics. An Introductory Treatise on the Principles of Dynamics, with applications and numerous examples, by A. E. H. Love, M.A., D.Sc., F.R.S. Second edition, revised. Demy 8vo. 12s

Geometrical Conics. By F. S. Macaulay, M.A., D.Sc. Second edition. Crown 8vo. 4s 6d

The Calculus for Beginners. By J. W. Mercer, M.A. Crown 8vo. 6s. **Exercises** only. 3s

Mathematical Problem Papers. Compiled and arranged by the Rev. E. M. Radford, M.A. Crown 8vo. 4s 6d net

A Treatise on Analytical Statics. With illustrations taken from the theories of Electricity and Magnetism. By E. J. Routh, M.A., Sc.D. Demy 8vo. Second edition. Vol. I. 14s. Vol. II. 14s

Radio-active Substances and their Radiations. By Sir Ernest Rutherford, D.Sc., LL.D., F.R.S. Demy 8vo. 15s net.

Advanced Exercises in Practical Physics. By Arthur Schuster, Ph.D., Sc.D., F.R.S., and Charles H. Lees, D.Sc. Second edition, revised. Demy 8vo. 8s

Practical Measurements. By A. W. Siddons, M.A., and A. Vassall, M.A. Fcap. 4to. 1s 6d

Elements of the Mathematical Theory of Electricity and Magnetism. By Sir J. J. Thomson, O.M., D.Sc., LL.D., Ph.D., F.R.S. Fourth edition. Crown 8vo. 10s

A Laboratory Note-Book of Elementary Practical Physics. Pt. I. Mechanics and Hydrostatics. Pt. II. Heat and Optics. Pt. III. Electricity and Magnetism. By L. R. Wilberforce, M.A., and T. C. Fitzpatrick, M.A. Crown 4to. 1s each

Papers set in the Mathematical Tripos, Part I, 1908—1912. Crown 8vo. 2s 6d net

Page 321-22 missing Jan. 31/41. RM